17.1 Lagrange Multipliers

The function we want to maximize is the negative squared distance from the point to the line (which is the same as minimizing the squared distance):

$$f(x,y) = -(x_0 - x)^2 - (y_0 - y)^2 \tag{1}$$

subject to the constraint that x and y are on the line:

$$g(x,y) = y - mx - b = 0 \tag{2}$$

We can then introduce the Lagrange multiplier λ and write the Lagrangian function:

$$\Lambda(x,y,\lambda) = \nabla f(x,y) - \lambda \nabla g(x,y) = 0 \tag{3}$$

which gives us:

$$\langle 2x_0 - 2x, 2y_0 - 2y \rangle = \lambda \langle -m, 1 \rangle \tag{4}$$

This equality and the original constraint yield the following system of three equations and three unknowns:

$$\begin{cases} 2x_0 - 2x = -\lambda m \\ 2y_0 - 2y = \lambda \\ y - mx - b = 0 \end{cases} \tag{5}$$

Solving the system, we obtain for $\langle x, y \rangle$:

$$\langle x, y \rangle = \frac{1}{1+m^2} \langle x_0 + m(y_0 - b), b + m(x_0 + my_0) \rangle \tag{6}$$

and for the Lagrange multiplier λ:

$$\lambda = -\frac{2(b + ax_0 - y_0)}{1+m^2} \tag{7}$$