Driven damped harmonic oscillator

Answer (a)

Firstly, the motion will be undamped if there is no friction, i.e., there is no conversion of mechanical energy to heat.

The motion will further be simple harmonic if the potential minimum can be well approximated by a parabola for small displacements.

Specifically, suppose a particle at position x is located in a potential field $V(x)$. Suppose further that the potential minimum is at x_0. Expanding the potential energy in a Taylor series about x_0, we have

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \cdots$$

Firstly, $V(x_0)$ can be set to zero, since we are only interested in differences in potential energy. Secondly, since x_0 is a potential minimum, we have $V'(x_0) = 0$ by definition (assuming $V(x)$ is continuously differentiable). So, defining $\bar{x} = x - x_0$ and $k = V''(x_0)$, the above equation becomes

$$V(\bar{x}) = \frac{1}{2}k\bar{x}^2.$$

This is exactly the equation for potential energy leading to simple harmonic motion.

Answer (b)

To solve the homogeneous equation

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

we try a solution of the form

$$x(t) = \exp\lambda t.$$

Inserting this into the homogeneous equation, factoring out $\exp\lambda t$ and noting that $\exp\lambda t \neq 0$ for any t, we deduce a quadratic equation for λ

$$m\lambda^2 + \gamma\lambda + k = 0$$

with solutions

$$\lambda_1 = \frac{\left(-\gamma - \sqrt{\gamma^2 - 4mk}\right)}{2m}, \qquad \lambda_2 = \frac{\left(-\gamma + \sqrt{\gamma^2 - 4mk}\right)}{2m}$$

Hence the homogeneous solution is a linear combination of the corresponding trial solutions

$$x(t) = A_1\exp(\lambda_1 t) + A_2\exp(\lambda_2 t),$$

where A_1 and A_2 are arbitrary constants.

Answer (c)

We are looking for a particular solution of the form

$$x(t) = H(\omega)\exp(i\omega t) \tag{0.1}$$

where $H(\omega)$ is the (complex) amplitude. Inserting this into the original equation (with the forcing term $e^{i\omega t}$) and solving for $A(\omega)$, we obtain

$$H(\omega) = \frac{1}{-m\omega^2 + i\omega\gamma + k} \tag{0.2}$$

The magnitude is defined as $G(\omega) = |H(\omega)|$ and the phase as $\phi(\omega) = \arg H(\omega)$. They can be plotted with the following Matlab code:

```
omega=linspace(0.01,100,2001);

m=1; k=1; gamma=0.1;

H=1./(-m*omega.^2+1i*omega*gamma+k);

figure(1)
plot(omega,abs(H))
set(gca,'XScale','log'); grid
xlabel('\omega'); ylabel('G(\omega)')

figure(2)
plot(omega,angle(H))
set(gca,'XScale','log','YTick',[-pi -pi/2 0]); grid
xlabel('\omega'); ylabel('\phi(\omega)')
```

This yields the figures below.

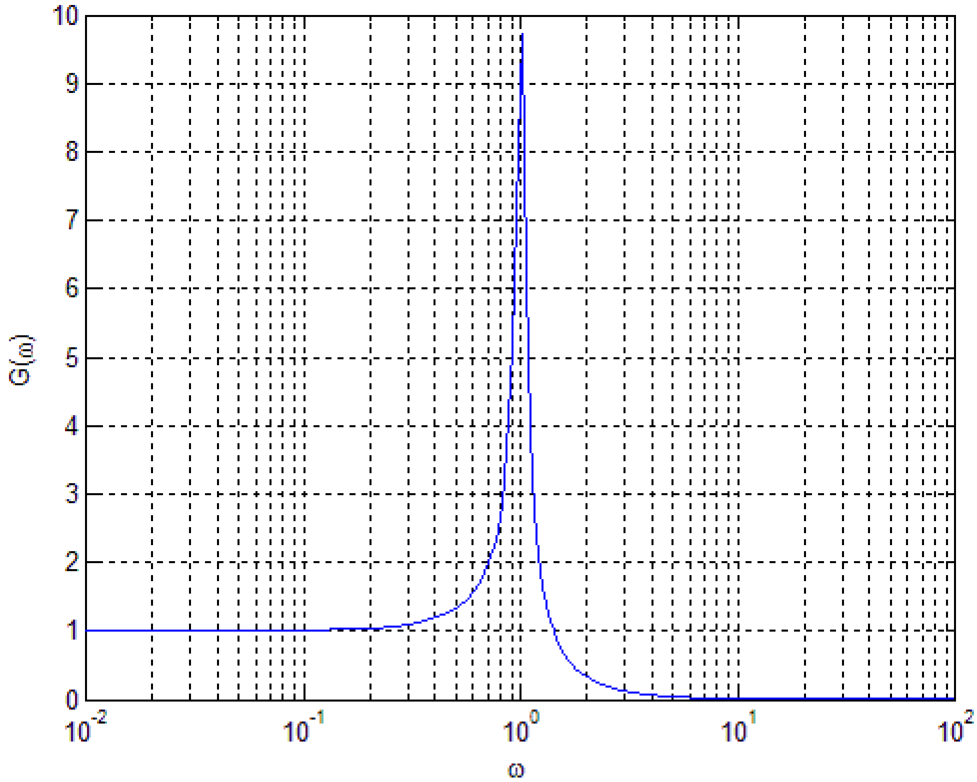

Figure 1 Amplitude response.

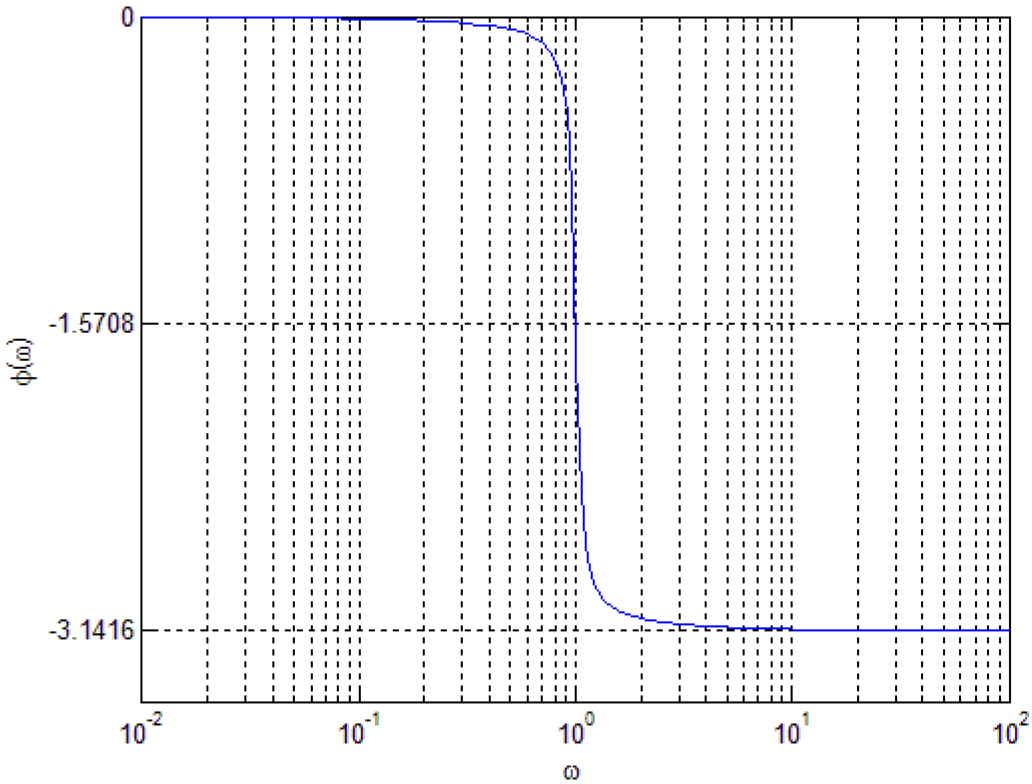

Figure 2 Phase response.

Answer (d)

To work out the first definition, we set

Answer (e)

Defining the Laplace transform of $x(t)$ as $X(s)$ and applying the standard theorems of Laplace transforms, (2.57) becomes

$$m[s^2X(s) - sx(0) - x'(0)] + \gamma[sX(s) - x(0)] + kX(s) = \frac{1}{s - i\omega}.$$

Using the initial conditions $x(0) = 0$ and $x'(0) = 0$, we find

$$X(s) = \frac{1}{(s - i\omega)(ms^2 + \gamma s + k)}. \qquad (0.3)$$

The solution can then be found using inverse Laplace transformation. We have used the following Maple commands (γ was replaced by b, since γ is protected in Maple)

```
> with(inttrans)
        [addtable, fourier, fouriercos, fouriersin, hankel, hilbert, invfourier,
         invhilbert, invlaplace, invmellin, laplace, mellin, savetable]

>
assume(omega, real, m, real, m, positive, b, real, b, positive, k, real, k,
    positive)
```

$$invlaplace\left(\frac{1}{(s - I\cdot\text{omega})\cdot(m\cdot s^2 + b\cdot s + k)}, s, t\right)$$

$$-\frac{I\cosh\left(\frac{1}{2}\frac{t\sqrt{b\sim^2 - 4\,m\sim k\sim}}{m\sim}\right)e^{-\frac{1}{2}\frac{t\,b\sim}{m\sim}}}{-I\,m\sim\omega\sim^2 - b\sim\omega\sim + I\,k\sim} + \frac{\cos(\omega\sim t) + I\sin(\omega\sim t)}{-m\sim\omega\sim^2 + I\,b\sim\omega\sim + k\sim} - \frac{I\,e^{-\frac{1}{2}\frac{t\,b\sim}{m\sim}}\sinh\left(\frac{1}{2}\frac{t\sqrt{b\sim^2 - 4\,m\sim k\sim}}{m\sim}\right)(b\sim + 2\,I\,m\sim\omega\sim)}{(-I\,m\sim\omega\sim^2 - b\sim\omega\sim + I\,k\sim)\sqrt{b\sim^2 - 4\,m\sim k\sim}}$$

The latter expression represents $x(t)$, with b replaced by γ.

Answer (f)

The lowest order correction is obtained by adding a term to the Taylor series expansion:

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \frac{1}{6}V'''(x_0)(x - x_0)^3$$

Using the results from (a) and defining $l \equiv V'''(x_0)$, we obtain

$$V(\bar{x}) = \frac{1}{2}k\bar{x}^2 + \frac{1}{6}l\bar{x}^3 \qquad (0.4)$$

The resulting force is

$$F = -\frac{dV}{dx} = -\frac{dV}{d\bar{x}} = -k\bar{x} - \frac{1}{2}l\bar{x}^2 \tag{0.5}$$

The last term is the desired correction term. Using $x_0 = 0$ so that $\bar{x} = x$, the modified forced damped oscillator equation becomes

$$m\ddot{x} + \gamma x + kx + \frac{1}{2}lx^2 = e^{i\omega t}. \tag{0.6}$$