MAS 864

Dhaval Adjodah (1)
dhaval@mit.edu

3.1 $m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega t}$

a) Undamped : $\gamma = 0$

b) $m\ddot{x} + \gamma\dot{x} + kx = 0$; homogeneous case

Let $x = e^{rt}$

$\Rightarrow$ Characteristic eq is $r^2 + \gamma r + k = 0$

$$\Rightarrow r = -\frac{\gamma}{2a} \pm \frac{\sqrt{\gamma^2 - 4k}}{2a} = r_1, r_2$$

$\therefore$ Sol$^{\underline{n}}$ is $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

c) $m = k = 1$; $\gamma = 0.1$

$\Rightarrow \ddot{x} + 0.1\dot{x} + x = e^{i\omega t}$

let $x = A e^{i(\omega t - \delta)}$ where δ is the phase lag

$\Rightarrow \{-(\omega)^2 + 0.1 i(\omega) + 1\} A e^{i(\omega t - \delta)} = e^{i\omega t}$

$\Rightarrow -(\omega)^2 + 0.1 i(\omega) + 1 = \dfrac{e^{i\delta}}{A}$

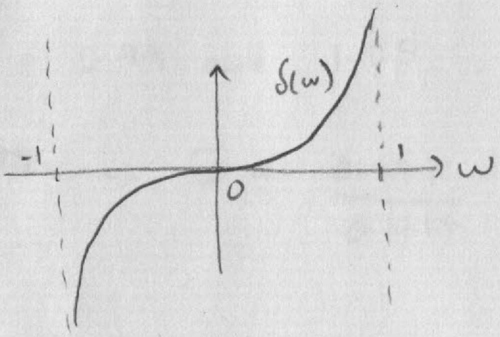
$\Rightarrow (1 - \omega^2)A + i(0.1\omega)A = e^{i\delta}$

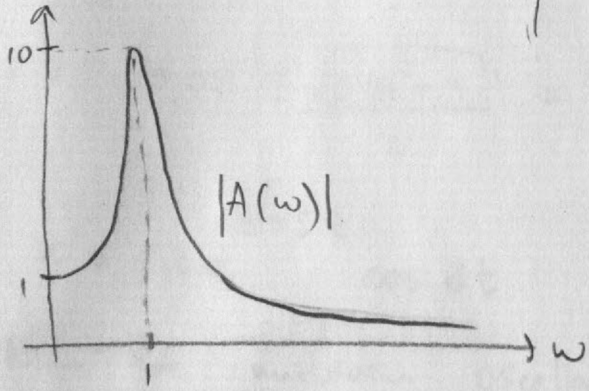
$$\Rightarrow A = \frac{e^{i\delta}}{(1-\omega^2) + i(0.1\omega)}$$

$$\text{Amp} = |A| = \frac{1}{\left[(1-\omega^2)^2 + (0.1\omega)^2\right]^{1/2}}$$

phase, $\tan\delta = \dfrac{0.1\omega}{1-\omega^2}$

$\Rightarrow \quad \delta = \tan^{-1}\left(\dfrac{0.1\omega}{1-\omega^2}\right)$





d) $A = \dfrac{1}{\left[(1-\omega^2)^2 + (0.1\omega)^2\right]^{1/2}}$

$\overline{E} = \frac{1}{2} k A^2 \sim \dfrac{1}{(1-\omega^2)^2 + (0.1\omega)^2}$

$\dfrac{dE}{d\omega} = 0$ when $\dfrac{d}{d\omega}\left((1-\omega^2)^2 + (0.1\omega)^2\right) = 0$

$\Rightarrow -2(1-\omega^2)2\omega + 2(0.1\omega)0.1 = 0$

$\Rightarrow -4\omega(1-\omega^2) + \frac{2}{100}(\omega) = 0$

$\Rightarrow -4\omega\left(1-\omega^2 + \frac{2}{100}\right) = 0$

$\Rightarrow \omega = 0$ or $\omega = \pm\sqrt{\frac{\cancel{96}}{100}} \simeq 0.997$

$\Rightarrow A_{max} = 89.286$

$A_{max}/2 = 44.64$

for $\dfrac{1}{(1-\omega^2)^2 + (0.1\omega)^2} = 44.64$

$\Rightarrow (1-\omega^2)^2 + 0.01\omega^2 = \dfrac{1}{44.64} \Rightarrow \omega^4 - 1.99\omega^2 + \left(1 - \dfrac{1}{44.64}\right) = 0$

$w^2 = 0.884,\ 1.106 \Rightarrow x = 0.94$ and 1.05

$\Rightarrow$ width $= 0.1119 \Rightarrow Q = \dfrac{0.997}{0.1119} = 8.92 \approx 9$

For undriven oscillator,

$$r = -0.05 \pm \frac{\sqrt{0.01^2 - 4}}{2} = -0.05 \pm i\alpha \quad \text{where } \alpha = \frac{\sqrt{4 - 0.01^2}}{2}$$

$$\Rightarrow x = A e^{-0.05t} \cos \alpha t$$

Now for undriven oscillator, $\gamma/2 = 0.05$

$\Rightarrow \gamma = 0.1$

(?) $\Rightarrow Q = \dfrac{w_0}{\gamma} = \dfrac{\alpha}{0.1} \approx 10$. Difference in numbers??

Hence, these two deffinitions are very similar.

γ is time for energy to decay to $1/e$ of value

$$\Rightarrow Q = \frac{w_0}{\gamma}$$

$$\frac{1}{\gamma} = \frac{Q}{w_0} = \frac{10^9}{100} = 10^7 \text{ s}.$$

$E = E_0 e^{-\gamma t}$

$t = 0,\ E = E_0$

$t = t_e,\ E = \dfrac{E_0}{e} e^{-\gamma t_e}$

$\Rightarrow -\gamma t_e = \ln\left(\dfrac{E_0 \frac{1}{e}}{E_0}\right)$

$t_e = \dfrac{\ln(e)}{\gamma} = 1/\gamma$

One cycle is $T = \dfrac{2\pi}{w} = \dfrac{2\pi}{\alpha}$

One radian is $\dfrac{1}{\alpha}$

$$\Rightarrow Q = \frac{E_0}{(E_0 - E_0 e^{-\gamma/\alpha})} = \frac{1}{1 - e^{-\gamma/\alpha}}$$

$\boxed{\approx 10.5}$

(3b)

Laplace Transform

$$\mathcal{L}(m\ddot{x} + \gamma\dot{x} + kx) = m\{s^2F(s) - s\,x(0) - \dot{x}(0)\} + \gamma\{sF(s) - x(0)\} + k\,F(s)$$

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s-i\omega}$$

$$\Rightarrow F(s)\left[s^2m + \gamma s + k\right] + \underbrace{x(0)\left[-ms - \gamma\right] + \dot{x}(0)\,m}_{=\,\alpha(s) = 0} = \frac{1}{s-i\omega}$$

the transient ??

Solving for steady state system,

$$F(s)\left[(s-\lambda_1)(s-\lambda_2)\right] = \frac{1}{s-i\omega}$$

where

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \beta \pm i\alpha$$

where $\alpha = \frac{\sqrt{3.99}}{2}$, $\beta = 0.05$

for $m, k = 1$; $\gamma = 0.1$

$$F(s) = \frac{1}{(s-i\omega)(s-\lambda_1)(s-\lambda_2)}$$

Assume $F(s) = \frac{A}{s-i\omega} + \frac{B}{s-\lambda_1} + \frac{C}{s-\lambda_2}$

$$= \frac{A}{s-i\omega} + \frac{B}{s-\lambda_1} + \frac{C}{s-\lambda_2}$$

$$\Rightarrow A(s-\lambda_1)(s-\lambda_2) + B(s-i\omega)(s-\lambda_2) + C(s-i\omega)(s-\lambda_1) = 1$$

$$\Rightarrow A(s^2 + s(-\lambda_1-\lambda_2) + \lambda_1\lambda_2) + B(s^2 + s(-i\omega-\lambda_2) + i\omega\lambda_2) + C(s^2 + s(-i\omega-\lambda_1) + i\omega\lambda_1) = 1$$

$$A + B + C = 0$$

$$A(-\lambda_1-\lambda_2) + B(-i\omega-\lambda_2) + C(-i\omega-\lambda_1) = 0$$

$$A\lambda_1\lambda_2 + B\,i\omega\lambda_2 + C\,i\omega\lambda_1 = 1$$

$\lambda_1 = \beta + i\alpha$

$\lambda_2 = \beta - i\alpha$

3c

$$\begin{bmatrix} 1 & 1 & 1 \\ -\lambda_1-\lambda_2 & -i\omega-\lambda_2 & -i\omega-\lambda_1 \\ \lambda_1\lambda_2 & i\omega\lambda_2 & i\omega\lambda_1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Using Mathematica,

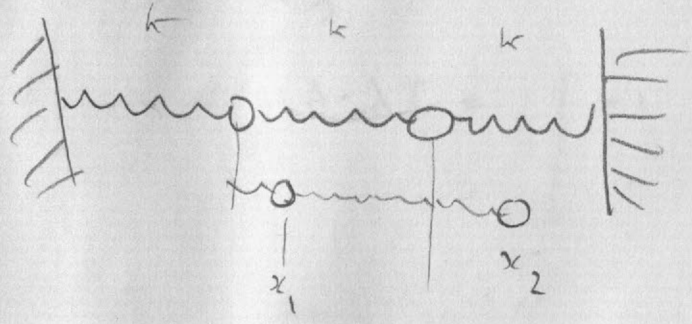
$$A = \frac{-1}{(\lambda_1 - i\omega)(-\lambda_2 + i\omega)} = \frac{-1}{-\lambda_1\lambda_2 + \omega^2 + i\omega(\lambda_1+\lambda_2)} = \frac{-1}{\beta^2+\alpha^2+\omega^2+i\alpha 2\beta}$$

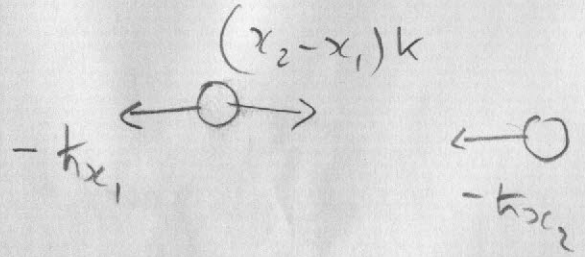
$$B = \frac{1}{(\lambda_1-\lambda_2)(\lambda_1 - i\omega)} = \frac{1}{2i\omega(\beta + i(\alpha-\omega))} = \frac{1}{-2\omega(\alpha-\omega) + i2\omega\beta}$$

$$C = \frac{-1}{(\lambda_1-\lambda_2)(\lambda_2 - i\omega)} = \frac{-1}{2i\omega(\beta - i(\alpha+\omega))} = \frac{-1}{2\omega(\alpha+\omega)+i2\omega\beta}$$

Hence Soln is $\mathcal{L}^{-1}(F(s))$

$$= A e^{i\omega t} + \underbrace{B e^{(\beta+i\alpha)t} + C e^{(\beta-i\alpha)t}}_{\text{transient ??}}$$





$$m\frac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1) = k(-2x_1 + x_2)$$

$$m\frac{d^2x_2}{dt^2} = -kx_2 - k(x_2 - x_1) = k(+x_1 - 2x_2)$$

$$\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{for} \quad \alpha = \frac{k}{m}$$

Now to find eigenvalues of matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

let $\det(A - \lambda I) = 0 \Rightarrow \det\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} = 0$

$\Rightarrow (-2-\lambda)^2 - 1 = 0$

$\Rightarrow 3 + \lambda^2 + 4\lambda = 0$

$\Rightarrow \lambda = -1 \text{ or } -3$

To find eigenvectors,

for $\lambda = -1 \Rightarrow \quad A - \lambda I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

for $\lambda = -3 \Rightarrow \quad A - \lambda I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$

$\Rightarrow$ eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence $\underline{\underline{soln}}$ is $\quad \vec{x} = a_1 e^{\sqrt{\lambda_1} t} \vec{V}_1 + a_2 e^{\sqrt{\lambda_2} t} \vec{V}_2$ where a_1 and a_2 are constants.

$= a_1 e^{it} \vec{v}_1 + a_2 e^{i\sqrt{3}t} \vec{v}_2$ $\qquad$ for $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= a_1 (\cos t + i \sin t) v_1 + a_2 (\cos \sqrt{3} t + i \sin \sqrt{3} t) v_2$

$= \left\{ a_1 \cos t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 \cos \sqrt{3} t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} + i \left\{ a_1 \sin t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 \cos \sqrt{3} t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$x_1 = a_1 \cos t + a_2 \cos \sqrt{3} t$

$x_2 = a_1 \cos t - a_2 \cos \sqrt{3} t$

$\Rightarrow \quad \vec{x} = a_1 \begin{bmatrix} \cos t \\ \cos t \end{bmatrix} + a_2 \begin{bmatrix} \cos \sqrt{3} t \\ -\cos \sqrt{3} t \end{bmatrix}$

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3

Z

$$y(k) = \alpha y(k-1) + (1-\alpha)x(k)$$

$$Z\{y(k)\} = Y(z) = Z\{\alpha y(k-1)\} + (1-\alpha) Z\{x(k)\}$$

$$= \alpha\left[\frac{Y(z)}{z} + y(-1)\right] + (1-\alpha)\left[X(z)\right]$$

$y(k<0) = 0$

$$\Rightarrow Y(z) - \frac{\alpha}{z}Y(z) = \cancel{\alpha y(-1)} + (1-\alpha)X(z)$$

$$\Rightarrow Y(z) = \frac{1-\alpha}{1-\alpha/z}X(z) = \underbrace{(1-\alpha)\frac{z}{z-\alpha}}_{H(z)} X(z)$$

$$\Rightarrow \cancel{Y(z) = H(z)X(z)}$$

$$\cancel{Z^{-1}\{Y(z)\} = Z^{-1}\{H(z)X(z)\}}$$

$$\Rightarrow y(k) = \cancel{\sum_{n=0}^{k} h(k-n)x(n) = \sum_{n=0}^{k} \delta(k-n)x(n)\,(1-\alpha)}$$

$$\cancel{= (1-\alpha)}$$

?

$$\sum_{r=0}^{\infty} r^n = \frac{1}{1-r} \qquad \cancel{(1-\alpha)\frac{1}{1-\alpha/z}X(z)}$$

$$(1-\alpha)\sum_{\ell=-\infty}^{\infty} x(\ell)x(k-\ell) = (1-\alpha)\sum_{\ell=0}^{k}\alpha^{\ell}x(k-\ell)$$

for $x(k) = e^{i\omega k}$

$$= (1-\alpha)\sum_{\ell=0}^{\infty}\alpha^{\ell}e^{i\omega(k-\ell)}$$

$$= (1-\alpha)e^{i\omega k}\frac{1}{1-\alpha e^{-i\omega}}$$

$$= \frac{1-\alpha}{1-\alpha e^{-i\omega}}e^{i\omega\ell}$$

$$y(k) = (\quad)\,x(k)$$