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$$\vec{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

7.1 a) $\langle e^{i\vec{k}\cdot\vec{x}} \rangle = \int e^{i\vec{k}\cdot\vec{x}} p(\vec{x})\, d\vec{x} = \sum_{n_1=0} \sum_{n_2=0} \cdots \sum_{n_3=0} \frac{(ik_1)^{n_1}}{n_1!} \langle x_1 \rangle^{n_1} \cdots \frac{(ik_N)^{n_N}}{n_N!} \langle x_N \rangle^{n_N}$

$$\exp\left(\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n \right) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle x^n \rangle$$

$$1 + \left(\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n \right) + \left(\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n \right)^2 \Big/ 2 + \dots = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle x^n \rangle$$

$$1 + \left(ikC_1 + \frac{(ik)^2}{2!} C_2 + \frac{(ik)^3}{3!} C_3 + \dots \right) + \frac{1}{2}\left(ikC_1 + \frac{(ik)^2}{2} C_2 + \frac{(ik)^3}{6!} C_3 + \dots \right)^2 + \frac{1}{6}\left(ikC_1 + \frac{(ik)^2}{6} C_2 + \frac{(ik)^3}{6!} C_3 + \dots \right)^3 + \dots = ik\langle x \rangle + \frac{(ik)^2}{2!}\langle x^2 \rangle + \frac{(ik)^3}{3!}\langle x^3 \rangle + \dots$$

Grouping by order of k,

for k, $ikC_1 = ik\langle x \rangle \Rightarrow C_1 = \langle x \rangle = \mu$, the mean of the distribution centered around zero

k^2, $\frac{(ik)^2 C_2}{2!} + \frac{(ik)^2 C_1}{2} = \frac{(ik)^2}{2!}\langle x^2 \rangle \Rightarrow \frac{C_2}{2} + \frac{C_1}{2} = \frac{\langle x^2 \rangle}{2} \Rightarrow C_2 = \langle x^2 \rangle - \langle x \rangle = \sigma^2 - \mu$

k^3, $\frac{(ik)^3}{3}$ ... $\langle x^3 \rangle \Rightarrow C_3 = \langle x^3 \rangle - \langle x \rangle$

Dimensionally wrong!

for k: $ikC_1 = ik\langle x \rangle \Rightarrow C_1 = \langle x \rangle = \mu$

k^2: $\frac{(ik)^2}{2!} C_2 + \frac{1}{2}(ikC_1)^2 = \frac{(ik)^2}{2!}\langle x^2 \rangle \Rightarrow C_2 + C_1^2 = \langle x^2 \rangle$

$\Rightarrow C_2 = \langle x^2 \rangle - \langle x \rangle^2 = ?$

k^3: $\frac{(ik)^3}{3!} C_3 + \frac{1}{2}\left((ik)^3 C_1 C_2 \right) + \frac{1}{6}\left(ikC_1 \right)^3 = \frac{(ik)^3}{3!}\langle x^3 \rangle$

$$C_3 + 3C_1C_2 + C_1^3 = \langle x^3 \rangle$$

$$\Rightarrow C_3 = \langle x^3 \rangle - 3\left\{ \langle x \rangle \left(\langle x^2 \rangle - \langle x \rangle^2 \right) \right\} - \langle x \rangle^3 = ?$$

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1 (b) For gaussian distribution, $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$,

Characteristic function is

$$\langle e^{ikx} \rangle = \int_{-\infty}^{\infty} e^{ikx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$\langle e^{ikx} \rangle = e^{i\mu k - \frac{1}{2}\sigma^2 k^2}$$

$$\log \langle e^{ikx} \rangle = \log e^{i\mu k - \frac{1}{2}\sigma^2 k^2} = i\mu k - \frac{1}{2}\sigma^2 k^2$$

Now $\log \langle e^{ikx} \rangle = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n$

$$= ik C_1 + \frac{(ik)^2}{2!} C_2 + \dots$$

$2\sigma^2$

$-x^2 + 2\mu x + ikx - \mu^2$

$= -(x^2 + (2\mu + ik)x + \mu^2)$

$= -\left\{\left(x - \frac{2\mu + ik}{2}\right)^2 + \mu^2 - \frac{(2\mu + ik)^2}{4}\right.$

Comparing coeff of k,

$$ikC_1 = i\mu k \Rightarrow C_1 = \mu$$

$$\frac{(ik)^2}{\cancel{2!}} C_2 = -\frac{1}{\cancel{2}}\sigma^2 k^2 \Rightarrow C_2 = \sigma^2$$

- relation btw cumulant and moment ?
- expansion of gaussian in terms of moments ?

Doing above integral, $\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{ikx - \frac{(x-\mu)^2}{2\sigma^2}} dx$

$$= \frac{e^{+\mu^2 - \frac{(2\mu + 2ik\sigma^2)^2}{4}}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x - \frac{2\mu + 2ik\sigma^2}{2}\right)^2} dx = \frac{e^{\mu^2 - (\mu + ik\sigma^2)^2}}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2} dx}_{=\sqrt{\pi}}$$

using $y = \frac{x - (\mu + ik\sigma^2)}{\sqrt{2\sigma^2}}$

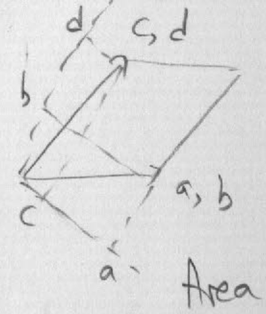
$$dy = \frac{dx}{\sqrt{2\sigma^2}}$$

1.2 (a) $\vec{y}(\vec{x}) = \begin{bmatrix} y_1(x_1, x_2) \\ y_2(x_1, x_2) \end{bmatrix}$ i.e. $\begin{bmatrix} y_1(\vec{x}) \\ y_2(\vec{x}) \end{bmatrix}$

We use the fact that $dy_1 dy_2 = \left| \mathrm{Det}\left(J(x_1, x_2)\right) \right| dx_1 dx_2$

$$J(x_1, x_2) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix}$$

$\frac{d}{dx_1} y(x + dx_1) = \left(\frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_1} \right)$



Area = ad-cb

$$\det J(x_1, x_2) = \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1}$$

Hence $dy_1 dy_2 = \left(\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1} \right) dx_1 dx_2$

b) $y_1 = \sqrt{-2\ln x_1}\, \sin x_2$

$y_2 = \sqrt{-2\ln x_1}\, \cos x_2$

Since $p(x_1, x_2)$ is uniform, $p(x_1, x_2) = C$

$$p(y_1, y_2) = D\{J(x_1, x_2)\} \overbrace{p(x_1, x_2)}^{C}$$

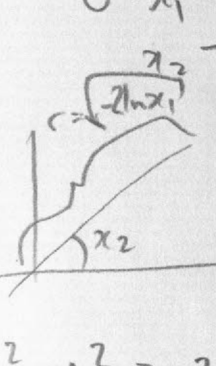
$$= D \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} C$$

$\frac{d}{dx_1}(-2\ln x_1)^{1/2}$

$= \frac{1}{2}(-2\ln x_1)^{-1/2} \cdot -2\frac{1}{x_1}$

$= \frac{-1}{x_1\sqrt{-2\ln x_1}}$

Imagine $x_1 \to r$
$x_2 \to \theta$



$$= D \begin{bmatrix} \frac{-1}{x_1\cancel{\sqrt{-2\ln x_1}}} \sin x_2 & \cancel{\sqrt{-2\ln x_1}} \cos x_2 \\ -\frac{1}{x_1\cancel{\sqrt{-2\ln x_1}}} \cos x_2 & \cancel{\sqrt{-2\ln x_1}} \sin x_2 \end{bmatrix} C$$

$y_1^2 + y_2^2 = -2\ln x_1$

$\Rightarrow x_1 = e^{-y_1^2 - y_2^2/2}$

$= e^{-r^2/2}$

$$= \cancel{C}\frac{\cancel{\sin^2 x_2} + \cancel{\cos^2 x_2}}{x_1}$$

$\frac{C\,1}{(\sin^2 + \cos^2) = 1}$

order 4 $\Rightarrow$ $x_n = x_{n-1} + x_{n-4}$

Start with any sequence other than 0000

1111010 $\rightarrow$

length of cycle in 2^{N-1}

Clock $= 10^{-9}$ $\Rightarrow N = 86.$

age of universe : π^s is nano-century

$$\log\left(2^{N-1}\right) \approx N = \log\left(\underset{\text{years}}{10^{10}} \times 10^{9} \underset{\text{1 GHz}}{\text{cycles/sec}}\right)$$

$\Rightarrow N \approx 86.$

7.4 a) $\dfrac{\partial p}{\partial t} = \underbrace{\dfrac{\langle \delta^2 \rangle}{2\tau}}_{D} \dfrac{\partial^2 p}{dt^2}$ for $p = p(x,t)$

Taking Fourier Transform on

<u>LHS</u>:

$$\mathcal{F}\left\{\frac{\partial p}{\partial t}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} \frac{\partial p(x,t)}{\partial t}\, dx = \frac{1}{2\pi}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{-ikx} p(x,t)\, dx$$

$$= \frac{1}{2\pi}\frac{\partial \hat{p}(k,t)}{\partial t}$$ where k is the wave number.

<u>RHS</u>:

$$\mathcal{F}\left\{\frac{\partial m}{\partial x}\right\} = \int_{-\infty}^{\infty} \overset{dv}{e^{-ikx}} \overset{u}{\frac{\partial p}{\partial x}(x,t)}\, dx$$

$$= \cancel{\left[e^{-ikx} p(x,t)\right]_{-\infty}^{\infty}}^{=0} - \int (-ki)\, e^{-ikx} p(x,t)\, dx$$

$$= ik\,\hat{m}(k,t)$$

$$\therefore \mathcal{F}\left\{\frac{\partial^2 p}{\partial x^2}\right\} = ik\,\mathcal{F}\left\{\frac{\partial p}{\partial x}\right\} = -k^2\,\hat{p}(k,t)$$

Hence, in Fourier space,

$$\frac{\partial \hat{p}}{\partial t} = -Dk^2\hat{p}$$

$$\Rightarrow \hat{p}(k,t) = \hat{p}_0\, e^{-k^2Dt}$$

↳ initial Fourier condition

For $p(x,0) = \delta(x)$

$$\Rightarrow \hat{p}_0 = \mathcal{F}\{\delta(x)\} = 1$$

$$\Rightarrow \hat{p}(k,t) = e^{-k^2Dt}$$

$$p(x,t) = \mathcal{F}^{-1}\{\hat{p}(k,t)\} = \mathcal{F}^{-1}\{e^{-k^2Dt}\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-k^2Dt}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2Dt + ikx}\, dk$$

Completing square on $-k^2Dt + ikx$

we get $-Dt\left(k^2 - \frac{ix}{Dt}k\right) = -Dt\left(k - \frac{ix}{2Dt}\right)^2 + Dt\left(\frac{ix}{2Dt}\right)^2$

$$\Rightarrow p(x,t) = \frac{e^{Dt\left(\frac{ix}{2Dt}\right)^2}}{2\pi}\int_{-\infty}^{\infty} e^{-Dt\left(k - \frac{ix}{2Dt}\right)^2}\, dk$$

For $l = \left(k - \frac{ix}{2Dt}\right)\sqrt{Dt}$

$dl = dk\sqrt{Dt}$

$$\Rightarrow p(x,t) = \frac{e^{-Dt\left(\frac{x}{2Dt}\right)^2}}{2\pi\sqrt{Dt}} \underbrace{\int_{-\infty}^{\infty} e^{-l^2}\, dl}_{\sqrt{\pi}} = \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

b) Comparing with the general gaussian $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

$\Rightarrow$ Variance $2\pi\sigma^2 = 4\pi Dt$

$\Rightarrow \sigma^2(t) = 2Dt$

c) For Brownian motion, $\frac{d^2\langle x^2\rangle}{dt^2} = -\frac{3\pi\mu A}{m}\frac{d\langle x^2\rangle}{dt}$

$\Rightarrow D = \frac{m}{3\pi\mu A}$

Hence $D \propto \frac{1}{\mu}$ inversely proportional

$\langle x^2\rangle = \sigma^2 = \frac{kT}{3\pi\mu a} t = 2Dt$

$\Rightarrow D = \frac{kT}{6\pi\mu a}$ $\qquad D = \frac{\langle \delta^2\rangle}{2\tau}$

$\boxed{D = \frac{1}{2}}$ $\qquad \sigma^2 = 2Dt = t$

$\sigma = \sqrt{t}$