(8.1) 1D Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

A. Finite diff approx.

$$\left(\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2}\right) = v^2 \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right)$$

$$u_j^{n+1} - 2u_j^n + u_j^{n-1} = \frac{v^2 \Delta t^2}{\Delta x^2} \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \left(\frac{v \Delta t}{\Delta x}\right)^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

B. second order in time + space

C. find the mode amplitudes

ansatz $\Rightarrow u_j^n = A(k)^n e^{ijk\Delta x}$

$$A^{n+1} e^{ijk\Delta x} = 2A^n e^{ijk\Delta x} - A^{n-1} e^{ijk\Delta x} + \left(\frac{v\Delta t}{\Delta x}\right)^2 \left(A^n e^{i(j+1)k\Delta x} - 2A^n e^{ijk\Delta x} + A^n e^{i(j-1)k\Delta x}\right)$$

$\longrightarrow$

$$Ae^{ijk\Delta x} = 2e^{ijk\Delta x} - \frac{1}{A}e^{ijk\Delta x} + \left(\frac{v\Delta t}{\Delta x}\right)^2 \left(e^{i(j+1)k\Delta x} - 2e^{ijk\Delta x} + e^{i(j-1)k\Delta x}\right)$$

$$A = 2 - 1/A + \left(\frac{v\Delta t}{\Delta x}\right)^2 \left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

$$A + 1/A = 2 - 2\left(\frac{v\Delta t}{\Delta x}\right)^2 + \left(\frac{v\Delta x}{\Delta x}\right)^2 2\cos(k\Delta x)$$

$$\frac{A^2+1}{A} = 2 + 2\left(\frac{v\Delta t}{\Delta x}\right)^2 \left(\cos(k\Delta x) - 1\right)$$

~~F~~. Attached

6. with damping:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial}{\partial t} \frac{\partial^2 u}{\partial x^2}$$

4. Attached

Finite diff approx:

$$\left(\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2}\right) = v^2\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right)$$

$$+ \gamma \frac{\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} - \frac{u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}}{\Delta x^2}\right)}{\Delta t}$$

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \frac{v^2 \Delta x^2}{\Delta t^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$+ \frac{\gamma \Delta t}{\Delta x^2}\left[\left(u_{j+1}^n - u_{j+1}^{n-1}\right) - 2\left(u_j^n - u_j^{n-1}\right) + \left(u_{j-1}^n - u_{j-1}^{n-1}\right)\right]$$

8.2 finite difference (explicit)

$$u_j^{n+1} = u_j^n + \frac{D\Delta t}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

Sim attached

A. $\Delta t = 0.1$ stable

$\Delta t = 0.5$ stable w/ high freq oscillations

$\Delta t = 1$ unstable

$$u_j^n = A(k)^n e^{ijk\Delta x}$$

$$A^{\cancel{n}+1} \cancel{e^{ijk\Delta x}} = \cancel{A^n}\cancel{e^{ijk\Delta x}} + \frac{D\Delta t}{\Delta x^2}\left(\cancel{A^n} e^{i(\cancel{j}+1)k\Delta x} - 2\cancel{A^n}\cancel{e^{ijk\Delta x}} + \cancel{A^n} e^{i(\cancel{j}-1)k\Delta x}\right)$$

$$A = 1 + \frac{D\Delta t}{\Delta x^2}\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)$$

$$A = 1 + D\frac{\Delta t}{\Delta x^2}\left(2\cos k\Delta x - 2\right)$$

$$A = 1 + \frac{2D\Delta t}{\Delta x^2}\underbrace{\left(\underbrace{\cos k\Delta x}_{\text{between } -1 \text{ and } 1} - 1\right)}_{\text{between } -2 \text{ and } 0}$$

8.3 Attached

8.4 $u_{j,k}^{n+1} = (1-\alpha)u_{j,k}^{n} +$

$\frac{\alpha}{4}\left(u_{j+1,k}^{n} + u_{j-1,k}^{n+1} + u_{j,k+1}^{n} + u_{j,k-1}^{n+1} \right)$

Demo Attached