6.1 A. Work out the first three cumulants C_1, C_2, C_3

$$e^{\left(\sum_{n=1}^{\infty} (ik)^n/n!\, C_n\right)} = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle x^n \rangle$$

$$e^x = 1 + x + x^2/2 + x^3/6 \ldots$$

$$1 + \left(\sum_{n=1}^{\infty} (ik)^n/n!\, C_n\right) + \frac{\left(\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n\right)^2}{2} + \frac{\left(\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n\right)^3}{6} \ldots$$

$$\sum_{n=1}^{\infty} (ik)^n/n!\, C_n = ikC_1 + \frac{(ik)^2}{2} C_2 + \frac{(ik)^3}{6} C_3 \ldots$$

$$= 1 + ikC_1 + \frac{(ik)^2}{2} C_2 + \frac{(ik)^3}{6} C_3$$

$$+ \frac{\left(ikC_1 + \frac{(ik)^2}{2} C_2 + \frac{(ik)^3}{6} C_3\right)^2}{2}$$

$$+ \frac{\left(ikC_1 + \frac{(ik)^2}{2} C_2 + \frac{(ik)^3}{6} C_3\right)^3}{6}$$

group by powers of k:

$$ikC_1 = ik\langle x \rangle \Rightarrow \boxed{C_1 = \langle x \rangle \text{ mean}}$$

$$\frac{(ik)^2}{2} C_2 + \frac{(ik)^2}{2} C_1^2 = \frac{(ik)^2}{2} \langle x^2 \rangle$$

$$\frac{(ik)^2}{2} C_2 = \frac{(ik)^2}{2} \langle x^2 \rangle - \frac{(ik)^2}{2} C_1^2$$

$$C_2 = \langle x^2 \rangle - C_1^2$$

$$C_2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$\boxed{C_2 = \sigma^2 \quad \text{variance}}$$

$$\frac{(ik)^3}{6} C_3 + \frac{(ik)^3 C_1^3}{6} + \frac{(ik)^3 C_1 C_2}{2} = \frac{(ik)^3}{6} \langle x^3 \rangle$$

$$C_3 + C_1^3 + 3 C_1 C_2 = \langle x^3 \rangle$$

$$C_3 = \langle x^3 \rangle - \langle x \rangle^3 - 3 \langle x \rangle \sigma^2$$

$$C_3 = \langle x^3 \rangle - \langle x \rangle^3 - 3 \langle x \rangle \left(\langle x^2 \rangle - \langle x \rangle^2 \right)$$

$$\boxed{C_3 = \langle x^3 \rangle + 2 \langle x \rangle^3 - 3 \langle x \rangle \langle x^2 \rangle}$$

B. Evaluate first three cumulants for a Gaussian dist.

$$p(x) = 1/\sqrt{2\pi\sigma^2}\, e^{-(x-\bar{x})^2/2\sigma^2}$$

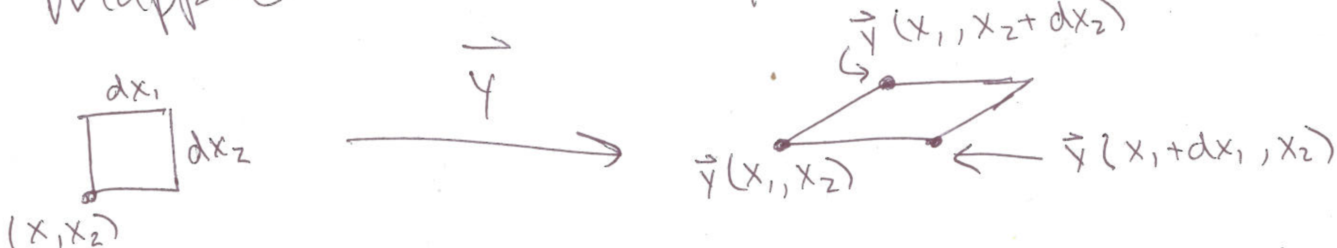
$$C_1 = \bar{x}$$

$$C_2 = \sigma^2$$

$$C_3 = \langle p(x)^3 \rangle + 2\bar{x}^3 - 3\bar{x}\sigma^2$$

I'm trying to evaluate $\langle p(x)^3 \rangle$ in sympy, but having trouble

6.2 A. $\vec{y}(\vec{x}) = (y_1(x_1, x_2), y_2(x_1, x_2))$ is a coordinate transformation what is the area of a differential element $dx_1 dx_2$ after it is mapped to the $\vec{y}$ plane?



$$b = \vec{y}(x_1+dx_1, x_2) - \vec{y}(x_1, x_2)$$

$$h = \vec{y}(x_1, x_2+dx_2) - \vec{y}(x_1, x_2)$$

assuming small dx_1, dx_2

$$b = \frac{\partial \vec{y}}{\partial x_1} dx_1$$

$$h = \frac{\partial \vec{y}}{\partial x_2} dx_2$$

$$\text{area} = dx_1 dx_2 \frac{\partial \vec{y}}{\partial x_1} \frac{\partial \vec{y}}{\partial x_2}$$

$$= dx_1 dx_2 \left(\frac{\partial y_1}{\partial x_1} + \frac{\partial y_2}{\partial x_1}\right)\left(\frac{\partial y_1}{\partial x_2} + \frac{\partial y_2}{\partial x_2}\right)$$

B Let $y_1 = \sqrt{-2\ln x_1}\,\sin(x_2)$

$y_2 = \sqrt{-2\ln x_1}\,\cos(x_2)$

if $p(x_1, x_2)$ is uniform, what is $p(y_1, y_2)$?

c. Write a uniform random number generator and transform it by eq 6.78 Numerically evaluate the first three cumulants of its output.

See Notebook

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code for random number generator in

amandaghassaei/RandomSystems/LFSR.js

6.3 A. order 4 maximal LSFR

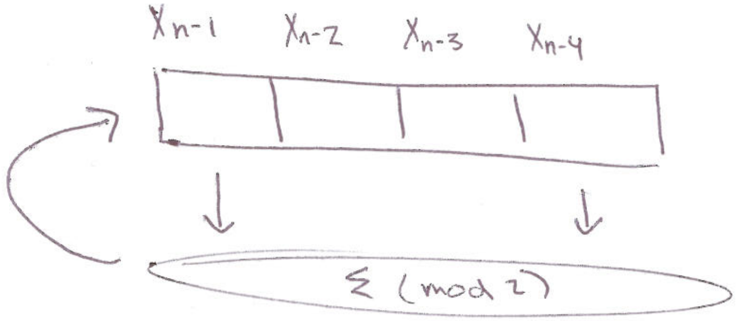
$$x_n \sum_{i=1}^{M} a_i x_{n-i} \pmod 2$$

let $M = 4$

for $i \in \{1, 4\}$ $a_i = 1$

else $a_i = 0$

$$x_n = (x_{n-1} + x_{n-4}) \pmod 2$$



0	0	0	1	$\Sigma \pmod 2 = 1$
1	0	0	0	$\Sigma \pmod 2 = 1$
1	1	0	0	$= 1$
1	1	1	0	$= 1$
1	1	1	1	$= 0$
0	1	1	1	$= 1$
1	0	1	1	$= 0$
0	1	0	1	$= 1$

B. If an LFSR has a clock rate of 1GHz how long must the register be for the time between repeats to be the age of the universe ($\sim 10^{10}$ years)

repeat time is $2^M - 1$ for register w/ M ~~steps~~ bits

assuming one step of LFSR per clock cycle:

$$(2^m - 1)(1/10^9) = 10^{10} * \underbrace{3.1 \times 10^7}_{\text{sec/year}}$$

$$2^m - 1 = 10^{26} \times 3.1$$

$$2^m = 3.1 \times 10^{26} + 1$$

$$M = \log_2 (3.1 \times 10^{26})$$

$$M \cong 88 \text{ bits}$$

6.4 A. Use a Fourier Transform to solve the diffusion eq:

$$\partial P/\partial t = D\, \partial^2 P/\partial x^2$$

assume initial cond. is normalized delta function. @ the origin.

$p(0) = 1$

$p(x) = 0$ for all $x \neq 0$

$$F\{p(x,t)\} = \hat{p}(w,t) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} p(x,t)\, e^{-iwx}\, dx$$

$$F\{\partial p/\partial t\} = \frac{\partial}{\partial t} F\{p(x,t)\}$$

$$F\{\partial^2 p/\partial x^2\} = -w^2 F\{p(x,t)\}$$

$$\frac{\partial}{\partial t} F\{p(x,t)\} = D(-w^2) F\{p(x,t)\}$$

$$\partial \hat{p}/\partial t = -w^2 D\, \hat{p}$$

this is solved by an equation of the form:

$\hat{p} = C e^{\lambda t}$ with $\lambda = -w^2 D$

$$\hat{p} = C e^{-w^2 D t}$$

given the initial conditions

$$p(x, 0) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{p}(\omega, 0) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} p(x,0) e^{-i\omega x} \, dx$$

$$= 1/\sqrt{2\pi} \left(p(0,0) e^{-i\omega(0)} \right)$$

$$= 1/\sqrt{2\pi}$$

$$1/\sqrt{2\pi} = C e^{-\omega^2 D(0)}$$

$$C = 1/\sqrt{2\pi}$$ is this right?

solution: $\hat{p} = 1/\sqrt{2\pi} \, e^{-\omega^2 D t}$

now we need to get this back to untransfomed space.

$$p(x,t) = F^{-1}\{\hat{p}(\omega, t)\}$$

given a gaussian:

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

the fourier transform is:

$$\hat{g}(\omega) = e^{-\omega^2 \sigma^2/2}$$

with $\sigma^2/2 = Dt$

we can ~~[illegible]~~ apply an inverse Fourier to $\hat{p}$:

$$F^{-1}\{\hat{p}\} = \frac{1}{\sqrt{2Dt}\,2\pi} e^{-x^2/4Dt}$$

$$\boxed{p(x,t) = \frac{1}{2\pi\sqrt{2Dt}} e^{-x^2/4Dt}}$$

B. what is the variance as a function of time?

$$\boxed{\sigma^2 = 2Dt}$$

c. How is the diffusion coeff. for Brownian motion related to the viscosity of a fluid?

$$\langle x^2 \rangle = {kT}/{3\pi \mu a}\ t$$

where μ is viscosity
a is diameter of particle

$$\langle x^2 \rangle = \langle x \rangle + \sigma^2$$

$$= \underbrace{\langle x \rangle}_{0} + 2Dt$$

$$2Dt = {kTt}/{3\pi \mu a}$$

$$\boxed{D = \frac{kT}{6\pi \mu A}}$$

D. Write a program for 10 random walkers

$D = \langle \delta^2 \rangle / 2\tau$

for Bernoulli

$\langle \delta^2 \rangle = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2$
$= 1$

$\tau = 1 \Rightarrow D = 1/2$

see notebook

E. What fraction of the trajectories should be contained in the error bars

3σ error =

$$\int_{-1.5t}^{1.5t} 1/\sqrt{2\pi t}\, e^{-x^2/2t}\, dx$$

evaled in notebook

87%