8.1 a.)

$$\frac{\partial^2 U}{\partial t^2} = v^2\left(\frac{U_j^{n+1} - 2U_j^n + U_{j-1}^n}{\Delta x^2}\right)$$

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{\Delta t^2} = v^2\left(\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}\right)$$

$$U_j^{n+1} = 2U_j^n - U_j^{n-1} + v^2\Delta t^2\left(\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}\right)$$

b.) second order approximation in space and time

(c.) $U_j^n = A(k)^n e^{ijk\Delta x}$

$$A^{n+1}e^{ijk\Delta x} = 2A^n e^{ijk\Delta x} - A^{n-1} e^{ijk\Delta x} + v^2\frac{\Delta t^2}{\Delta x^2}\left(A^n e^{i(j+1)k\Delta x} - 2A^n e^{ijk\Delta x} + A^n e^{i(j-1)k\Delta x}\right)$$

$$A^{n+1} = 2A^n - A^{n-1} + v^2\frac{\Delta t^2}{\Delta x^2}\left(A^n e^{ik\Delta x} - 2A^n + A^n e^{-ik\Delta x}\right)$$

$$A = 2 - \frac{1}{A} + v^2\frac{\Delta t^2}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

$$= 2 - \frac{1}{A} + \left(\frac{v\Delta t}{\Delta x}\right)^2 (2\cos k\Delta x - 2)$$

$$A^2 = 2A\left(1 + \left(\frac{v\Delta t}{\Delta x}\right)^2(\cos k\Delta x - 1)\right) - 1$$

$$A^2 \underbrace{- 2\left(1 + \left(\frac{v\Delta t}{\Delta x}\right)^2(\cos k\Delta x - 1)\right)}_{B} A + 1 = 0$$

$$A = \frac{-B \pm \sqrt{B^2 - 4}}{2}$$

d.) $\max|A^2| = \left(\frac{-B+\sqrt{B^2-4}}{2}\right)\left(\frac{-B-\sqrt{B^2-4}}{2}\right)$

$= \frac{\cancel{B^2} - \cancel{B^2} + 4}{4} = 1$

$\left(\frac{-B+\sqrt{B^2-4}}{2}\right)^2 = 1$

$B^2 - 2B\sqrt{B^2-4} + B^2 - 4 = 4$

$B^2 - B\sqrt{B^2-4} = 4$

$-B\sqrt{B^2-4} = 4 - B^2$

$B^2(B^2-4) = 16 - 8B^2 + B^4$

$\cancel{B^4} - 4B^2 = 16 - 8B^2 + \cancel{B^4}$

$4B^2 = 16$

$|B| = 2$

$B \geq -2 \longrightarrow$ for stability

$\frac{-B-\sqrt{B^2-4}}{2} = 1$

$B^2 + B\sqrt{B^2-4} = 4$

$|B| \leq 2$

$-2 \leq -2\left(1 + \left(\frac{r\Delta t}{\Delta x}\right)^2(\cos(k\Delta x) - 1)\right) \leq 2$

$0 \leq -2\left(r\frac{\Delta t}{\Delta x}\right)^2(\cos(k\Delta x) - 1) \leq 4$

$0 \leq -2(\cos(k\Delta x) - 1) \leq 4\left(\frac{\Delta x}{\Delta t r}\right)^2$

$-1 \leq \cos(k\Delta x) \leq 2\left(\frac{\Delta x}{\Delta t r}\right)^2 - 1$

$\cos(k\Delta x) \leq 1 \leq \underline{\underline{2\left(\frac{\Delta x}{\Delta t r}\right)^2}} - 1$

$1 \leq \left(\frac{\Delta x}{r\Delta t}\right)^2 \qquad \left(\frac{r\Delta t}{\Delta x}\right)^2 \leq 1$

e.) yes.

f.) 0 0 0 1 0 0 0 0

unstable $\Delta x = 1$ $\qquad \left(\frac{r\Delta t}{\Delta x}\right)^2 = 2$

$\Delta t = 1$

$r = \sqrt{2}$

$t=0$	0	0	0	1	0	0	0	0
$t=1$	0	0	2	-2	2	0	0	0
$t=2$	0	4	0	3	0	4	0	0
$t=3$	8	-8	12	8	12	-8	[illegible]	[illegible]
$t=4$	-16	52	-24	29	-24	52	-16	16

marginally stable

$\Delta t = 1 \quad \Delta x = 1 \quad \left(\frac{v \Delta t}{\Delta x}\right)^2 = 1$

$v = 1$

$t=0$	0	0	0	1	0	0	0	0
$t=1$	0	0	1	0	1	0	0	0
$t=2$	0	1	0	1	0	1	0	0
$t=3$	1	0	1	0	1	0	1	0
	0	1	0	1	0	1	0	1
	0	0	1	0	1	0	1	0
	0	0	0	1	0	1	0	0
	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	0	0

g.) $$v_j^{n+1} = 2v_j^n - v_j^{n-1} + r^2 \Delta t^2 \frac{(v_{j+1}^n - 2v_j^n + v_{j-1}^n)}{\Delta x^2} + r\frac{\Delta t}{\Delta x^2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n - v_{j+1}^{n-1} + 2v_j^{n-1} - v_{j-1}^{n-1})$$

$$e^{-\omega \Delta t i} = 2 - e^{i\omega\Delta t} + r^2 \frac{\Delta t^2}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) + r\frac{\Delta t}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x} - e^{ik\Delta x + i\omega\Delta t} + 2e^{i\omega\Delta t} - e^{-ik\Delta x + i\omega\Delta t}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = r^2 \frac{\partial^2 u}{\partial x^2} + r \frac{\partial}{\partial t}\frac{\partial u}{\partial x^2} \qquad u(x,t) = Ae^{i(kx-\omega t)}$$

$$(-\omega i)^2 A e^{i(kx-\omega t)} = r^2 (ki)^2 A e^{i(kx-\omega t)} + r(-\omega i)(ki)^2 A e^{i(kx-\omega t)}$$

for small r

$$-\omega^2 \cancel{A e^{i(kx-\omega t)}} = 0 + r\omega k^2 i \cancel{A e^{i(kx-\omega t)}}$$

$$\omega = -rk^2 i$$

$$\omega i = rk^2$$

The change in phase ~~w~~ corresponding to a k displacement in space is equal to a $-rk^2$ change in phase.

As $r\uparrow$ the same change in phase corresponds to increasingly small changes in space, until everything breaks down for discretized approximations.

8.2) $$V_j^{n+1} = V_j^n + \frac{\Delta t\, V^2}{\Delta x^2} \left(V_{j+1}^n - 2V_j^n + V_{j-1}^n \right)$$

Stable for $\Delta x < \frac{1}{2}$, however implicit solution stable for all.

8.3)

8.4). As lattice size ↑, larger a helps, since ↑ a ↑ convergence rate. Unfortunately results are best with small a ($a \leq 0.5$). Stable for $a \leq 2$.