Problem 3.1

Consider the motion of a damped, driven harmonic oscillator:

$$m\ddot{x} + \gamma\dot{x} + Kx = e^{i\omega t}$$

a.) Under what conditions will the governing equations for small displacements of a particle around an arbitrary 1D potential minimum be simple undamped harmonic motion?

With the current system, simple harmonic motion will occur if:

1.) There is no damping $\gamma = 0$

2.) There is no forcing function

This would result in an ODE like:

$$m\ddot{x} + kx = 0$$

b.) Find the solution to the homogeneous equation and comment on the possible cases. How does the amplitude depend on the frequency.

homogenious equation:

$$m\ddot{x} + \gamma\dot{x} + Kx = 0 \quad [1]$$

Guess the solution $x = e^{rt}$, $\dot{x} = re^{rt}$, $\ddot{x} = r^2e^{rt}$, Sub in to eq [1]

$$mr^2e^{rt} + \gamma re^{rt} + ke^{rt}$$

Divide out e^{rt}:

$$mr^2 + \gamma r + k = 0 \quad [2] \text{ characteristic eq.}$$

$$r^2 + \frac{\gamma r}{m} + \frac{k}{m} = 0$$

Continued

Continuing problem 3.1.b)

Solving for roots of r

$$r^2 + \frac{\gamma r}{m} + \frac{k}{m} = 0$$

$$r_{1,2} = \frac{-\frac{\gamma}{m} \pm \left(\left(\frac{\gamma}{m}\right)^2 - 4\frac{k}{m}\right)^{1/2}}{2} = \frac{-\gamma}{2m} \pm \left[\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}\right]^{1/2}$$

Sub in $\omega = \sqrt{k/m}$

$$r_1 = -\frac{\gamma}{2m} + \left[\left(\frac{\gamma}{2m}\right)^2 - \omega^2\right]^{1/2} \qquad r_2 = -\frac{\gamma}{2m} - \left[\left(\frac{\gamma}{2m}\right)^2 - \omega^2\right]^{1/2}$$

Pluging into the guessed equation

$$\boxed{x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}} \qquad [3]$$

With this system there are 3 possible cases:

Case 1: Underdamped where $\left(\frac{\gamma}{2m}\right)^2 < \omega^2$

Case 2: critically damped where $\left(\frac{\gamma}{2m}\right)^2 = \omega^2$

Case 3: over damped where $\left(\frac{\gamma}{2m}\right)^2 > \omega^2$

Additionally, the amplitudes (C_1, C_2) are not interacting with the frequencey & thus are not affecting it.

Problem 3.1. c)

Find the particlular solution to the inhomogeneous problem by assuming a response at the driving frequency, and plot its magnitude and phase as a function of the driving frequency for $m = k = 1$, $\gamma = 0.1$

inhomogenious equation $m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega t}$

Solving for the particular Solution

assuming $x = Ae^{i\omega t}$, $\dot{x} = Ai\omega e^{i\omega t}$, $\ddot{x} = A(-1)\omega^2 e^{i\omega t}$

Pluging in the guesses for x

$$m(-A\omega^2 e^{i\omega t}) + \gamma A i\omega e^{i\omega t} + Ae^{i\omega t} k = e^{i\omega t}$$

$$-mA\omega^2 + \gamma A i\omega + Ak = 1$$

$$A = \frac{1}{\gamma i\omega - m\omega^2 + k}$$

$$\boxed{X_p(t) = \frac{e^{i\omega t}}{\gamma i\omega - m\omega^2 + k}} \qquad \boxed{\phi = \operatorname{atan}\left(\frac{\gamma\omega}{-m\omega^2 + k}\right)}$$

Where $A = \frac{1}{r} = \frac{1}{\left((\gamma i\omega)^2 + (-m\omega^2 + k)^2\right)^{1/2}}$

$$\boxed{A = \frac{1}{\left[(\gamma\omega)^2 + (-m\omega^2 + k)^2\right]^{1/2}}}$$

Problem 3.1

Part c continued

Define Variables

```
m = 1; %mass
k = 1; %stiffness
y = .1;%damping
w = linspace(-10,10,2000); %Frequency Rads/s
```

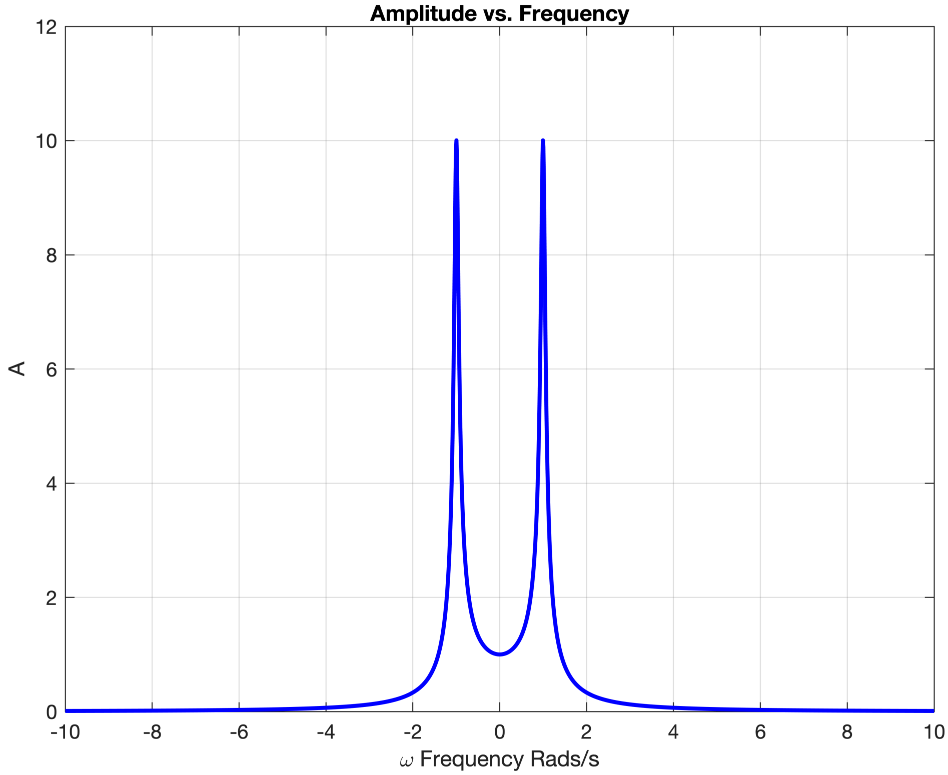
Define Equations of Amplitude and Phase as fuctions of ω

```
A = 1./((y.*w).^2 +(-m.*w.^2+k).^2).^.5;
phi = atan2(-y.*w,(-m.*w.^2+k));
```

Plot equations

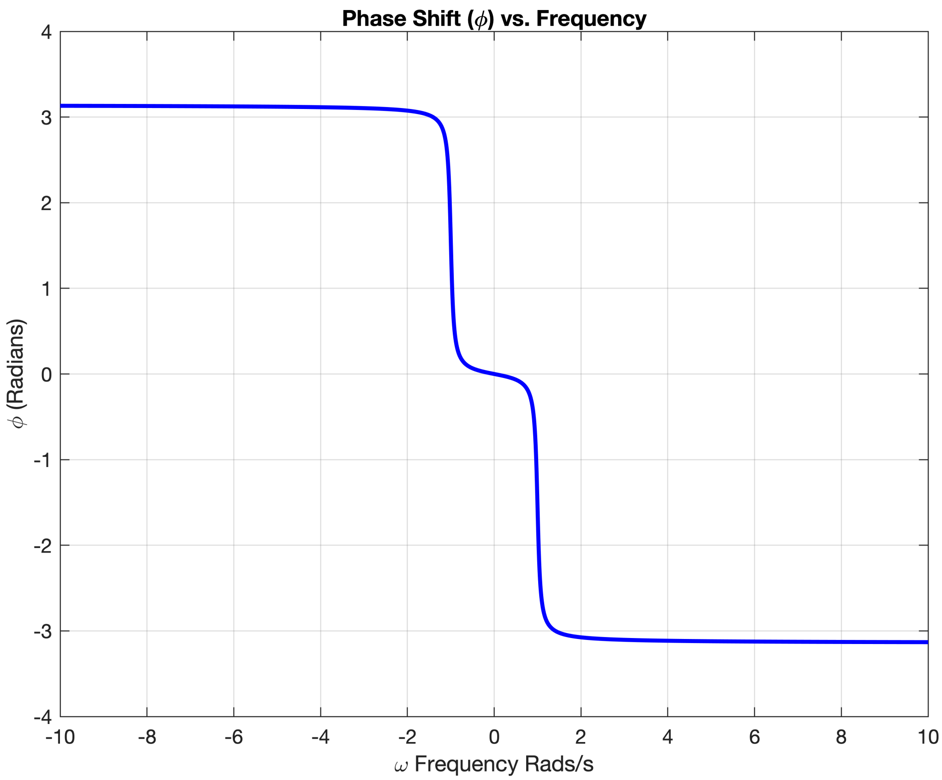
Plot the Amplitude as a function of frequency ω

```
figure(1)
axis([-10,10,-1,12])
plot(w,A,'b',"LineWidth",2)
title('Amplitude vs. Frequency')
ylabel('A')
xlabel('\omega Frequency Rads/s')
grid on
```



Plot the Phase as a function of frequency ω

```
figure(2)
axis([-10,10,-4,4])
plot(w,phi,'b',"LineWidth",2)
title('Phase Shift (\phi) vs. Frequency')
ylabel('\phi (Radians)')
xlabel('\omega Frequency Rads/s')
grid on
```


Phase Shift (ϕ) vs. Frequency
ϕ (Radians)
ω Frequency Rads/s
4
3
2
1
0
-1
-2
-3
-4
-10
-8
-6
-4
-2
0
2
4
6
8
10