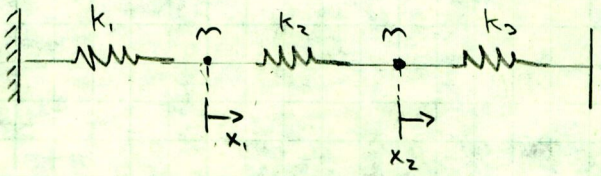
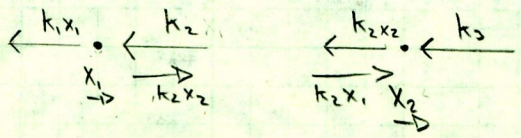
Problem 3.2

Explicitly Solve the system of differential equations for two coupled harmonic oscillators and find the normal modes by matrix diagonalization.



free body diagram:



$\Sigma F = ma$

mass 1

$$\ddot{x}_1 m_1 = -x_1 k_1 - x_1 k_2 + k_2 x_2$$

$$\ddot{x}_1 m_1 + x_1(k_1 + k_2) - x_2 k_2 = 0$$

mass 2

$$\ddot{x}_2 m_2 = -x_2 k_2 - x_2 k_3 + x_1 k_2$$

$$\ddot{x}_2 m_2 + x_2(k_2 + k_3) - x_1 k_2 = 0$$

EOMS:

$$\ddot{x}_1 + x_1\left(\frac{k_1 + k_2}{m}\right) - x_2 \frac{k_2}{m} = 0$$

$$\ddot{x}_2 + x_2\left(\frac{k_2 + k_3}{m}\right) - x_1 \frac{k_2}{m} = 0$$

$k_1 = k_2 = k_3$

Put into matrix form $\frac{d^2\vec{x}}{dt^2} + A\vec{x} = 0$

Coupled EOM's

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{bmatrix} = 0$$

$$A = \begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{bmatrix}$$

Continuing problem 3.2

Solve for eigen values of A $\quad |I\vec{\lambda} - A| = 0$

$$\begin{vmatrix} \lambda - \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \lambda - \frac{2k}{m} \end{vmatrix} = \left(\lambda - \frac{2k}{m}\right)^2 - \left(\frac{k}{m}\right)^2$$

$$\lambda^2 + 4\left(\frac{k}{m}\right)^2 - 2\frac{k}{m}\lambda - 2\frac{k}{m}\lambda - \left(\frac{k}{m}\right)^2 = 0$$

$$\lambda^2 - 4\frac{k}{m}\lambda + 3\left(\frac{k}{m}\right)^2 = 0$$

$$\lambda_{1,2} = \frac{+4\frac{k}{m} \pm \left(\left(-4\frac{k}{m}\right)^2 - 4\left(+3\left(\frac{k}{m}\right)^2\right)\right)^{1/2}}{2} = \frac{2k}{m} \pm \left(\left(\frac{2k}{m}\right)^2 - 3\left(\frac{k}{m}\right)^2\right)^{1/2}$$

$$\lambda_{1,2} = \frac{2k}{m} \pm \frac{k}{m}$$

eigen values

$$\boxed{\lambda_1 = \frac{3k}{m} \qquad \lambda_2 = \frac{k}{m}}$$

Plug back into A to find eigen vectors

eigen vectors

$$\lambda_1 \Rightarrow \begin{bmatrix} \frac{3k}{m} - \frac{2k}{m} & -k/m \\ -k/m & \frac{3k}{m} - \frac{2k}{m} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 \Rightarrow \begin{bmatrix} \frac{k}{m} - \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} - \frac{2k}{m} \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$