*

a) write down the straightforward finite-difference approximation.

let's $u(x+\Delta x, t) = u(x,t) + \Delta x \cdot \frac{du}{dx} + \frac{1}{2!} \cdot \Delta x^2 \cdot \frac{d^2u}{dx^2} + O[(\Delta x)^3]$

(8.t) --- $\frac{u^n_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} + O[(\Delta x)^2] = \frac{d^2 u}{dx^2}$

$$\frac{1}{\Delta t}\left[\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} - \frac{u(x,t) - u(x, t-\Delta t)}{\Delta t}\right]$$

$$= \frac{1}{\Delta t^2} \cdot \left(u(x, t+\Delta t) - 2u(x,t) + u(x, t-\Delta t)\right) = \frac{d^2 u}{dt^2} + O[(\Delta t)^2]$$

$$\therefore \quad \frac{d^2 u}{dx^2} = \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{(\Delta x)^2} + O[(\Delta t)^2]$$

$$\frac{d^2 u}{dt^2} = \frac{u^{n+1}_j - 2u^n_j + u^{n-1}_j}{(\Delta t)^2} + O[(\Delta t)^2]$$

$$\therefore \quad v^2 \cdot \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{\Delta x^2} + O[(\Delta x)^2] = \frac{u^{n+1}_j - 2u^n_j + u^{n-1}_j}{(\Delta t)^2} + O[(\Delta t)^2]$$

8.1.a

$$\therefore \quad u^{n+1}_j = v^2 \cdot \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{\Delta x^2} \cdot (\Delta t)^2 + O[(\Delta x)^2] \cdot (\Delta t)^2 + 2u^n_j - u^{n-1}_j + O[(\Delta t)^4]$$

$$= v^2 \frac{(\Delta t)^2}{(\Delta x)^2} \left[u^n_{j+1} - 2u^n_j + u^n_{j-1}\right] + 2u^n_j - u^{n-1}_j + O[(\Delta x)^2 (\Delta t)^2] + O[(\Delta t)^4]$$

8.1.b

② → Second order in both time / space

~~8~~ use von Neumann stability criterion to find the mode of A.

8.1.c

$$u_j^n = A(k)^n e^{ijk\Delta x}$$

from 8.1.a $\quad u_j^{n+1} = v^2 \cdot \frac{(\Delta t)^2}{(\Delta x)^2}\left\{u_{j+1}^n - 2u_j^n + u_{j-1}^n\right\} + 2u_j^n - u_j^{n-1}$

$$A(k)^{n+1} \cdot e^{ijk\Delta x} = v^2 \cdot \left(\frac{\Delta t}{\Delta x}\right)^2 \left\{A(k)^n \left\{e^{i(j+1)k\Delta x} - 2e^{ijk\Delta x} + e^{i(j-1)k\Delta x}\right\}\right\}$$

$$+ A(k)^{n-1}\left(2A(k) \cdot e^{ijk\Delta x} - e^{ijk\Delta x}\right)$$

$$A(k)^2 = v^2 \cdot \left(\frac{\Delta t}{\Delta x}\right)^2 \left\{\cancel{A(k)}\, e^{ik\Delta x} - 2 + e^{i(-1)k\Delta x}\right\}\} + (2A(k) - 1).$$

$$\cancel{A(k)} \quad A^2 = \left(v^2 \frac{\Delta t^2}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) + 2\right) A - 1$$

$$A^2 - \underbrace{\left(v^2 \frac{\Delta t^2}{\Delta x^2} \cdot (2\cos k\Delta x - 2)\right)}_{b} A + 1 = 0$$

$$A = \frac{v^2 \frac{\Delta t^2}{\Delta x^2} \cdot (\cos k\Delta x - 1) \pm \sqrt{\left(v^2 \frac{\Delta t^2}{\Delta x^2} \cdot (\cos k\Delta x - 1)\right)^2 - 1}}{1}$$

Q to be stable. $\quad |A|^2 \quad (b + \sqrt{b^2-1})(b \cancel{-} \sqrt{b^2-1}) \quad \forall.$

8.1.d

$$= (b \pm \sqrt{b^2-1})^2 \leq 1$$

$\therefore \quad \cancel{b^2 \pm 2b^2-1}$

$$b^2 + 2b\sqrt{b^2-1} + b^2 - 1 \leq 1$$

$$2b^2 + 2b\sqrt{b^2-1} - 1 \leq 1$$

$$\therefore \quad b^2 - (b^2 - 1) \leq 1$$

$$1 \leq 1$$

8.1.d continue

$b \to$ 2 options. $A = b + \sqrt{b^2-1}$, $b - \sqrt{b^2-1}$

to be stable, $|A|^2$ $(b+\sqrt{b^2-1})^2 \leq 1$ or $(b-\sqrt{b^2-1})^2 \leq 1$

$2b^2-1+2b\sqrt{b^2-1} \leq 1$

let's $2b^2-1+2b\sqrt{b^2-1} = 1$

$2b^2-2 = -2b\sqrt{b^2-1}$

$b^2-1 = -b\sqrt{b^2-1}$

$\cancel{b^4}-2b^2+1 = b^2(b^2-1)$

$= \cancel{b^4}-b^2$

$b^2-1 \leq b^2$ $1-b^2 \leq 0$

$-b^2+1 \leq 0$

$1 \leq b^2$ $\boxed{|b|\leq 1}$

to be stable, ~~$(1-b)(1+b) \leq 0$~~

$b \geq 1$ or $b \leq -1$

$b^2 - 2b\sqrt{b^2-1} + b^2 - 1 \leq 1$

$2b^2 \cancel{-1} - 2b\sqrt{b^2-1} \leq \cancel{1}$

$b^2-1 \leq b\sqrt{b^2-1}$

$\cancel{b^4}-2b^2+1 \leq \cancel{b^4}-b^2$

$-b^2+1 \leq 0$

$\therefore 1 \leq b^2$

$\to b \geq 1$, $b \leq -1$.

$\therefore b = v^2\left(\frac{\Delta t}{\Delta x}\right)^2 \cdot (\cos k\Delta x - 1)$ ~~≥ 1 or ≤ -1~~ ≤ 1

$b = 1 + v^2\left(\frac{\Delta t}{\Delta x}\right)^2 \cdot (\cos k\Delta x - 1) \geq 1$

$v^2\left(\frac{\Delta t}{\Delta x}\right)^2 \cdot \underline{(\cos k\Delta x - 1)} \geq 0$ ... ①.

$\underline{v^2 \left(\frac{\Delta t}{\Delta x}\right)^2} \cdot (\cos k\Delta x - 1) \leq -2$ ... ②

① $\to$ $\cos k\Delta x \geq 1$

$-2 \leq v^2\left(\frac{\Delta t}{\Delta x}\right)^2 \cdot (\cos k\Delta x - 1) \leq 0$

②

$0 \leq \left(\frac{v\Delta t}{\Delta x}\right)^2 (\cos k\Delta x - 1) \leq 1$

$1 \leq (\cos kx) \leq \left(\frac{\Delta x}{v\Delta t}\right)^2 + 1$

g) If the equation is replaced by

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + r \frac{\partial}{\partial t} \frac{\partial^2 u}{\partial x^2}$$

assume that

$$u(x,t) = Ae^{i(kx-\omega t)}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(-\omega i \cdot Ae^{i(kx-\omega t)}\right)$$

$$= -\omega^2 \cdot Ae^{i(kx-\omega t)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = -k^2 \cdot Ae^{i(kx-\omega t)}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) = -k^2 \cdot -\omega i \cdot Ae^{i(kx-\omega t)}$$

$$\therefore -\omega^2 \cdot \cancel{\boxed{A}} = v^2 \cdot -k^2 \cdot \cancel{\boxed{A}} + r \cdot -k^2 \cdot -\omega i \cancel{\boxed{A}}$$

$$-\omega^2 = v^2 \cdot -k^2 + r \cdot k^2 \omega i$$

$$r k^2 \omega i = v^2 k^2 - \omega^2$$

$$\underline{\omega = \pm kv - \frac{k^2 i}{2} r.}$$

$$0 = \omega^2 + r k^2 i \omega - k^2 v^2$$

$$\omega = \frac{-(rk^2 i) \pm \sqrt{(rk^2 i)^2 + 4k^2 v^2}}{2}$$

$$\underline{\omega} =$$

$$= \frac{-rk^2 i \pm \sqrt{-r^2 k^4 + 4k^2 v^2}}{2.}$$

Since r is small

$$\doteqdot \frac{-rk^2 i}{2} \pm \underline{kV}.$$