INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 24, 359–373 (1987)

THE METHOD OF MOVING ASYMPTOTES—A NEW METHOD FOR STRUCTURAL OPTIMIZATION

KRISTER SVANBERG

Optimization and Systems Theory, Royal Institute of Technology, S-10044 Stockholm, Sweden

SUMMARY

A new method for non-linear programming in general and structural optimization in particular is presented. In each step of the iterative process, a strictly convex approximating subproblem is generated and solved. The generation of these subproblems is controlled by so called 'moving asymptotes', which may both stabilize and speed up the convergence of the general process.

1. INTRODUCTION

In this paper a new method for structural optimization is presented. The method, which is called the 'method of moving asymptotes' and MMA, is based on a special type of convex approximation.

Ideally, a method for structural optimization should be flexible and general. It should be able to handle not only element sizes as design variables, but also, for instance, shape variables and material orientation angles. It should also be able to handle 'all kinds' of constraints, provided only that the derivatives of the constraint functions with respect to the design variables could be calculated (analytically or numerically). Thus, the method should be able to handle general non-linear programming problems. In addition, it should take into consideration the characteristics of structural optimization problems, e.g. usually very expensive function evaluations but still the possibility to calculate gradients. Further, the method should be 'stable' and generate a sequence of improved feasible (or almost feasible) solutions of the considered problem.

We hope, and believe, that these requirements and wishes are to a rather large extent met by the method of moving asymptotes. In addition, MMA is easy to implement and use.

The outline of the paper is as follows: in section 2 the method is presented in rather general terms, whereas a more technical description is given in section 3. In section 4 a dual method is suggested for solving the subproblems generated by the method. In section 5 it is shown how to avoid the unpleasant situation that a subproblem becomes infeasible. Finally, in section 6, some interesting numerical test results are presented.

2. GENERAL DESCRIPTION OF THE METHOD

Consider a structural optimization problem of the following general form:

P: minimize $\quad f_0(\mathbf{x}) \quad (\mathbf{x} \in R^n)$

subject to $\quad f_i(\mathbf{x}) \leqslant \hat{f}_i, \quad$ for $i = 1, \ldots, m$

0029-5981/87/020359-15$07.50

Received 3 October 1985
Revised 17 June 1986

and

$$\underline{x}_j \leqslant x_j \leqslant \bar{x}_j, \qquad \text{for } j = 1, \ldots, n$$

where $\mathbf{x} = (x_1, \ldots, x_n)^T$ is the vector of design variables, $f_0(\mathbf{x})$ is the objective function, typically the structural weight, $f_i(\mathbf{x}) \leqslant \hat{f}_i$ are behaviour constraints, typically limitations on stresses and displacements, $\underline{x}_j$ and $\bar{x}_j$ are given lower and upper bounds ('technological constraints') on the design variables.

A well established general approach for attacking such problems is to generate and solve a sequence of explicit subproblems according to the following iterative scheme:

Step 0. Choose a starting point $\mathbf{x}^{(0)}$, and let the iteration index $k = 0$.
Step I. Given an iteration point $\mathbf{x}^{(k)}$, calculate $f_i(\mathbf{x}^{(k)})$ and the gradients $\nabla f_i(\mathbf{x}^{(k)})$ for $i = 0, 1, \ldots, m$.
Step II. Generate a subproblem $P^{(k)}$ by replacing, in P, the (usually implicit) functions f_i by approximating explicit functions $f_i^{(k)}$, based on the calculations from step I.
Step III. Solve $P^{(k)}$ and let the optimal solution of this subproblem be the next iteration point $\mathbf{x}^{(k+1)}$. Let $k = k + 1$ and go to step I.

The process is interrupted when some convergence criteria are fulfilled, or simply when the user is satisfied with the current solution $\mathbf{x}^{(k)}$.

This general approach was suggested for element sizing problems already in Reference 1, where it was further suggested that, for $i > 0$, the approximating function $f_i^{(k)}$ should be obtained by a linearization (i.e. a first order Taylor expansion) in the reciprocal elemental sizes $(1/\mathbf{x}_j)$ of f_i at the current iteration point $\mathbf{x}^{(k)}$, while $f_0^{(k)}$ should be chosen identical to f_0.

A generalization of the method in Reference 1, to other structural optimization problems than just element sizing, was recently proposed in Reference 2, where it was suggested that each $f_i^{(k)}$ should be obtained by a linearization of f_i in 'mixed' variables; either x_j or $1/x_j$ dependent on the sign of the derivative $\partial f_i/\partial \mathbf{x}_j$ at $\mathbf{x}^{(k)}$.

The method suggested in this paper (MMA) may be interpreted as a further generalization of the method in Reference 2. In brief, each $f_i^{(k)}$ is obtained by a linearization of f_i in variables of the type $1/(x_j - L_j)$ or $1/(U_j - x_j)$ dependent on the signs of the derivatives of f_i at $\mathbf{x}^{(k)}$. The values of the parameters L_j and U_j are normally changed between the iterations, and we will sometimes refer to L_j and U_j as 'moving asymptotes'.

It may easily be shown that the method of Reference 2 is obtained as a special case of MMA by letting $L_j = 0$ and $U_j \to \infty$.

However, by permitting also other values of L_j and U_j, a more flexible and powerful method is obtained. In particular, as will be demonstrated on a simple example, these parameters can be used to efficiently stabilize the general approach described above.

It should perhaps also be noted that when MMA is used, the variables are not required to be non-negative. In fact, MMA is sensitive to neither translation nor scaling of the variables.

3. TECHNICAL DESCRIPTION OF THE METHOD

MMA follows the general approach (steps 0–III) described in the previous section. Thus, to define the method, it must be described:

(a) how the functions $f_i^{(k)}$ should be defined
(b) how the subproblem $P^{(k)}$ should be solved, given that the $f_i^{(k)}$ have been chosen.

We start with the first question, i.e. how to choose $f_i^{(k)}$. The question of how to solve $P^{(k)}$ is left to section 4.

Given the iteration point $\mathbf{x}^{(k)}$ (at iteration k), values of the parameters $L_j^{(k)}$ and $U_j^{(k)}$ are chosen, for $j=1,\ldots,n$, such that

$$L_j^{(k)} < x_j^{(k)} < U_j^{(k)} \tag{1}$$

Different rules for how to choose these values are discussed in detail later.

Then, for each $i=0,1,\ldots,m$, $f_i^{(k)}$ is defined by

$$f_i^{(k)}(\mathbf{x}) = r_i^{(k)} + \sum_{j=1}^{n}\left(\frac{p_{ij}^{(k)}}{U_j^{(k)}-x_j} + \frac{q_{ij}^{(k)}}{x_j - L_j^{(k)}}\right) \tag{2}$$

where

$$p_{ij}^{(k)} = \begin{cases} (U_j^{(k)} - x_j^{(k)})^2 \partial f_i/\partial x_j, & \text{if } \partial f_i/\partial x_j > 0 \\ 0, & \text{if } \partial f_i/\partial x_j \leqslant 0 \end{cases} \tag{3}$$

$$q_{ij}^{(k)} = \begin{cases} 0, & \text{if } \partial f_i/\partial x_j \geqslant 0 \\ -(x_j^{(k)} - L_j^{(k)})^2 \partial f_i/\partial x_j, & \text{if } \partial f_i/\partial x_j < 0 \end{cases} \tag{4}$$

$$r_i^{(k)} = f_i(\mathbf{x}^{(k)}) - \sum_{j=1}^{n}\left(\frac{p_{ij}^{(k)}}{U_j^{(k)}-x_j^{(k)}} + \frac{q_{ij}^{(k)}}{x_j^{(k)} - L_j^{(k)}}\right) \tag{5}$$

where all derivatives $\partial f_i/\partial x_j$ are evaluated at $\mathbf{x}=\mathbf{x}^{(k)}$.

Then, as is easily checked, $f_i^{(k)}$ is a first order approximation of f_i at $\mathbf{x}^{(k)}$, i.e.

$$f_i^{(k)}(\mathbf{x}^{(k)}) = f_i(\mathbf{x}^{(k)}) \quad \text{and} \quad \partial f_i^{(k)}/\partial x_j = \partial f_i/\partial x_j \text{ at } \mathbf{x}=\mathbf{x}^{(k)}$$

for $i=0,1,\ldots,m$ and $j=1,\ldots,n$.

Further, the second derivatives of $f_i^{(k)}$, at any point $\mathbf{x}$ such that $L_j^{(k)} < x_j < U_j^{(k)}$ for all j, are given by

$$\frac{\partial^2 f_i^{(k)}}{\partial x_j^2} = \frac{2p_{ij}^{(k)}}{(U_j^{(k)} - x_j)^3} + \frac{2q_{ij}^{(k)}}{(x_j - L_j^{(k)})^3} \tag{6}$$

and

$$\frac{\partial^2 f_i^{(k)}}{\partial x_j \partial x_l} = 0 \text{ if } j \neq l$$

Thus, since $p_{ij}^{(k)} \geqslant 0$ and $q_{ij}^{(k)} \geqslant 0$, $f_i^{(k)}$ is a convex function. In particular, at $\mathbf{x}=\mathbf{x}^{(k)}$

$$\frac{\partial^2 f_i^{(k)}}{\partial x_j^2} = \begin{cases} \dfrac{2\partial f_i/\partial x_j}{U_j^{(k)} - x_j^{(k)}}, & \text{if } \partial f_i/\partial x_j > 0 \\ -\dfrac{2\partial f_i/\partial x_j}{x_j^{(k)} - L_j^{(k)}}, & \text{if } \partial f_i/\partial x_j < 0 \end{cases} \tag{7}$$

Thus, the closer $L_j^{(k)}$ and $U_j^{(k)}$ are chosen to $x_j^{(k)}$, the larger become the second derivatives, the more curvature is given to the approximating functions $f_i^{(k)}$, and the more conservative becomes the approximation of the original problem. More precisely, the following holds.

Assume that $\bar{f}_i^{(k)}$ and $\bar{\bar{f}}_i^{(k)}$ are two different approximating functions corresponding to the parameter values $\{\bar{L}_j^{(k)}, \bar{U}_j^{(k)}\}$ and $\{\bar{\bar{L}}_j^{(k)}, \bar{\bar{U}}_j^{(k)}\}$, respectively. If it holds that $\bar{L}_j^{(k)} \leqslant \bar{\bar{L}}_j^{(k)} < x_j^{(k)} < \bar{\bar{U}}_j^{(k)} \leqslant \bar{U}_j^{(k)}$ for all j, then, for all points $\mathbf{x}$ such that $\bar{\bar{L}}_j^{(k)} < x_j < \bar{\bar{U}}_j^{(k)}$ for all j, it holds that $\bar{\bar{f}}_i^{(k)}(\mathbf{x}) \leqslant \bar{f}_i^{(k)}(\mathbf{x})$.

The proof of this statement, which easily follows by straightforward calculations, is omitted here.

Correspondingly, if $L^{(k)}$ and $U^{(k)}$ are chosen 'far away' from $\mathbf{x}^{(k)}$, then $f_i^{(k)}$ becomes close to linear, i.e. without any curvature for all reasonable $\mathbf{x}$, i.e. for those $\mathbf{x}$ which are close to $\mathbf{x}^{(k)}$ compared to

$\mathbf{L}^{(k)}$ and $\mathbf{U}^{(k)}$. In the extreme case that '$L_j^{(k)} = -\infty$' and '$U_j^{(k)} = +\infty$' for all j, then the $f_i^{(k)}$ become (in the limit as $L_j^{(k)} \rightarrow -\infty$ and $U_j^{(k)} \rightarrow +\infty$) identical to the linear functions

$$f_i^{(k)}(\mathbf{x}) = f_i(\mathbf{x}^{(k)}) + \sum_j (\partial f_i/\partial x_j)(x_j - x_j^{(k)})$$

which are the approximating functions used in the well-known 'sequence of linear programmes' method, which has been used for structural optimization in Reference (3), for instance.

If, instead, $L_j^{(k)} = 0$ and '$U_j^{(k)} = +\infty$' for all j, then the $f_i^{(k)}$ become (in the limit as $U_j^{(k)} \rightarrow \infty$) identical to the approximating functions used in the method suggested in Reference 2. In this case $f_i^{(k)}$ becomes linear in the variable x_j if $\partial f_i/\partial x_j \geqslant 0$ and strictly convex in x_j if $\partial f_i/\partial x_j < 0$.

In MMA, the values of $L_j^{(k)}$ and $U_j^{(k)}$ are always finite. Then each $f_i^{(k)}$ becomes strictly convex in all variables x_j except in those for which $\partial f_i/\partial x_j = 0$ at $x = x^{(k)}$. (If $\partial f_i/\partial x_j = 0$ at $\mathbf{x} = \mathbf{x}^{(k)}$ then $f_i^{(k)}$ becomes independent of x_j.) Now, with the approximating functions $f_i^{(k)}$ defined by (2), the following subproblem, called $P^{(k)}$, is obtained:

$P^{(k)}$: minimize

$$\sum_{j=1}^{n} \left(\frac{p_{oj}^{(k)}}{U_j^{(k)} - x_j} + \frac{q_{oj}^{(k)}}{x_j - L_j^{(k)}} \right) + r_o^{(k)}$$

subject to

$$\sum_{j=1}^{n} \left(\frac{p_{ij}^{(k)}}{U_j^{(k)} - x_j} + \frac{q_{ij}^{(k)}}{x_j - L_j^{(k)}} \right) + r_i^{(k)} \leqslant \hat{f}_i, \quad \text{for } i = 1, \ldots, m$$

and

$$\max\{\underline{x}_j, \alpha_j^{(k)}\} \leqslant x_j \leqslant \min\{\bar{x}_j, \beta_j^{(k)}\} \quad \text{for } j = 1, \ldots, n$$

Here, the parameters $\alpha_j^{(k)}$ and $\beta_j^{(k)}$ are 'move limits' which are probably not very crucial. However, to avoid the possibility of any unexpected 'division by zero' while solving the subproblem, $\alpha_j^{(k)}$ and $\beta_j^{(k)}$ should at least be chosen such that

$$L_j^{(k)} < \alpha_j^{(k)} < x_j^{(k)} < \beta_j^{(k)} < U_j^{(k)} \tag{8}$$

for example $\alpha_j^{(k)} = 0{\cdot}9 L_j^{(k)} + 0{\cdot}1 x_j^{(k)}$ and $\beta_j^{(k)} = 0{\cdot}9 U_j^{(k)} + 0{\cdot}1 x_j^{(k)}$.

It will now briefly be discussed how to choose values for the 'moving asymptotes' $L_j^{(k)}$ and $U_j^{(k)}$.

Provided that the given lower and upper bounds $\underline{x}_j$ and $\bar{x}_j$ on the variables are 'physically reasonable', a simple choice is to let

$$L_j^{(k)} = \underline{x}_j - s_0(\bar{x}_j - \underline{x}_j) \quad \text{and} \quad U_j^{(k)} = \bar{x}_j + s_0(\bar{x}_j - \underline{x}_j) \tag{9}$$

where s_0 is a fixed real number, say $s_0 = 0{\cdot}1$. Here, $L_j^{(k)}$ and $U_j^{(k)}$ do not depend on k, i.e. they are 'fixed asymptotes' rather than 'moving'.

An important special case of 'fixed asymptotes' occurs when x_j stands for the transverse size of an element (or a group of elements). Then it is often reasonable to let

$$L_j^{(k)} = 0 \quad \text{and} \quad U_j^{(k)} = \text{'a large number', say } 10\bar{x}_j \tag{10}$$

Even if the simplicity of 'fixed asymptotes' is appealing, we believe that in order to fully exploit the flexibility of MMA, one must permit the asymptotes to move in some clever way between the iterations.

A general (although heuristic) rule for how to change the values of $L_j^{(k)}$ and $U_j^{(k)}$ is the following:

(a) If the process tends to oscillate, then it needs to be stabilized. This stabilization may be accomplished by moving the asymptotes closer to the current iteration point.

(b) If, instead, the process is monotone and slow, it needs to be 'relaxed'. This may be accomplished by moving the asymptotes away from the current iteration point.

A simple implementation of this 'rule' is the following. Here, s is a given real number less than unity, e.g. $s = 0{\cdot}7$.

For $k=0$ and $k=1$, let

$$L_j^{(k)} = x_j^{(k)} - (\bar{x}_j - \underline{x}_j) \text{ and } U_j^{(k)} = x_j^{(k)} + (\bar{x}_j - \underline{x}_j) \tag{11}$$

For $k \geqslant 2$

(a) If the signs of $x_j^{(k)} - x_j^{(k-1)}$ and $x_j^{(k-1)} - x_j^{(k-2)}$ are opposite, indicating an oscillation in the variable x_j, then let

$$\begin{aligned} L_j^{(k)} &= x_j^{(k)} - s(x_j^{(k-1)} - L_j^{(k-1)}) \\ U_j^{(k)} &= x_j^{(k)} + s(U_j^{(k-1)} - x_j^{(k-1)}) \end{aligned} \tag{12}$$

(b) If the signs of $x_j^{(k)} - x_j^{(k-1)}$ and $x_j^{(k-1)} - x_j^{(k-2)}$ are equal, indicating that the asymptotes are slowing down the convergence in the variable x_j, then let

$$\begin{aligned} L_j^{(k)} &= x_j^{(k)} - (x_j^{(k-1)} - L_j^{(k-1)})/s \\ U_j^{(k)} &= x_j^{(k)} + (U_j^{(k-1)} - x_j^{(k-1)})/s \end{aligned} \tag{13}$$

There are, of course, a lot of possible alternatives to this implementation. One could, for example, refuse to 'relax' the asymptotes unless all three of $x_j^{(k)} - x_j^{(k-1)}, x_j^{(k-1)} - x_j^{(k-2)}$ and $x_j^{(k-2)} - x_j^{(k-3)}$ have equal signs, etc.

One could also use, for instance, $\sqrt{s}$ instead of s in (13), so that it needs two 'relaxations' of the asymptotes to fully compensate for one 'tightening'. (This makes the method more conservative and stable.)

We have, so far, not worked very hard on the question of how to choose values for $L_j^{(k)}$ and $U_j^{(k)}$. This is clearly a possible area for further investigations. However, even with the crude choices suggested above, (9)–(13), the method has indeed worked very well on different test problems.

4. A DUAL METHOD TO SOLVE THE SUBPROBLEMS

To simplify notation, we will in this section suppress the iteration index k on the coefficients in the subproblem. Further, we will write α_j instead of $\max\{\underline{x}_j, \alpha_j\}$, β_j instead of $\min\{\bar{x}_j, \beta_j\}$ and b_i instead of $\hat{f}_i - r_i$.

Then the subproblem $P^{(k)}$, defined in the previous section, may be written as follows:

$P^{(k)}$: minimize

$$\sum_{j=1}^{n}\left(\frac{p_{0j}}{U_j - x_j} + \frac{q_{0j}}{x_j - L_j}\right) + r_0$$

subject to

$$\sum_{j=1}^{n}\left(\frac{p_{ij}}{U_j - x_j} + \frac{q_{ij}}{x_j - L_j}\right) \leqslant b_i, \quad \text{for } i = 1, \ldots, m$$

and

$$\alpha_j \leqslant x_j \leqslant \beta_j, \quad \text{for } j = 1, \ldots, n$$

where $p_{ij} \geqslant 0, q_{ij} \geqslant 0$ and $L_j < \alpha_j \leqslant \beta_j < U_j$.

$P^{(k)}$ is a convex, separable problem. Therefore, a dual method analogous to the ones described in References 4 and 5 could be used for its solution. Such a method will here be described.

The Lagrangian function corresponding to $P^{(k)}$ is given by

$$l(x, y) = f_0^{(k)}(\mathbf{x}) + \sum_{i=1}^{m} y_i f_i^{(k)}(\mathbf{x})$$

which, after trivial calculations, equals

$$r_0 - \mathbf{y}^{\mathrm{T}}\mathbf{b} + \sum_{j=1}^{n} \left(\frac{p_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{p}_j}{U_j - x_j} + \frac{q_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{q}_j}{x_j - L_j} \right)$$

$$= \mathbf{r}_0 - \mathbf{y}^{\mathrm{T}}\mathbf{b} + \sum_{j=1}^{n} l_j(x_j, \mathbf{y})$$

where

$$\mathbf{b} = (b_1, \ldots, b_m)^{\mathrm{T}}, \quad \mathbf{p}_j = (\mathrm{p}_{1j}, \ldots, p_{mj})^{\mathrm{T}}, \quad \mathbf{q}_j = (q_{1j}, \ldots, q_{mj})^{\mathrm{T}},$$

$$\mathbf{y} = (y_1, \ldots, y_m)^{\mathrm{T}}$$

and

$$l_j(x_j, \mathbf{y}) = \frac{p_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{p}_j}{U_j - x_j} + \frac{q_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{q}_j}{x_j - L_j}$$

$\mathbf{y}$ is the vector of Lagrange multipliers or 'dual variables'.

Next, the 'dual objective function' W is defined, for $\mathbf{y} \geqslant 0$ (i.e. all $y_i \geqslant 0$), as follows:

$$W(\mathbf{y}) = \min_{\mathbf{x}} \{ l(\mathbf{x}, \mathbf{y}); \alpha_j \leqslant x_j \leqslant \beta_j \quad \text{for all } j \}$$

$$= r_0 - \mathbf{y}^{\mathrm{T}}\mathbf{b} + \sum_{j=1}^{n} W_j(\mathbf{y})$$

where

$$W_j(\mathbf{y}) = \min_{x_j} \{ l_j(x_j, \mathbf{y}); \alpha_j \leqslant x_j \leqslant \beta_j \} \tag{14}$$

It will now be shown that it is easy to write down, explicitly, the minimizing x_j in (14). This minimizing x_j, which clearly depends on $\mathbf{y}$, will be denoted by $x_j(\mathbf{y})$.

First note that since $\mathbf{y} \geqslant 0$ it follows that $p_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{p}_j \geqslant 0$ and $q_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{q}_j \geqslant 0$. Therefore, $l_j(x_j, \mathbf{y})$ is convex as a function of x_j. Next, note that in the rare case that $p_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{p}_j = 0$ and $q_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{q}_j = 0$ (i.e. $p_{0j} = 0, q_{0j} = 0, y_i p_{ij} = 0$ and $y_i q_{ij} = 0$ for all i) $l_j(x_j, \mathbf{y})$ does not depend on x_j at all. Thus, in this rare case, any x_j between α_j and β_j minimizes $l_j(x_j, \mathbf{y})$.

From now on, we may thus assume that at least one of $p_{0j} + \mathbf{y}^{\mathrm{T}} p_j$ or $q_{0j} + \mathbf{y}^{\mathrm{T}} q_j$ is strictly positive. Then the derivative of $l_j(x_j, \mathbf{y})$ with respect to x_j is given by

$$l_j'(x_j, \mathbf{y}) = \frac{p_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{p}_j}{(U_j - x_j)^2} - \frac{q_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{q}_j}{(x_j - L_j)^2} \tag{15}$$

and the second derivative of $l_j(x_j, \mathbf{y})$ with respect to x_j is given by

$$l_j''(x_j, \mathbf{y}) = \frac{2(p_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{p}_j)}{(U_j - x_j)^3} + \frac{2(q_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{q}_j)}{(x_j - L_j)^3} \tag{16}$$

Since $l_j''(x_j, \mathbf{y})$ is strictly positive, the derivative $l_j'(x_j, \mathbf{y})$ is strictly increasing in x_j, and we may draw the following conclusions concerning the minimizing x_j in (14) (denoted by $x_j(\mathbf{y})$):

1. If $l'_j(\alpha_j, \mathbf{y}) \geqslant 0$ then $x_j(\mathbf{y}) = \alpha_j$ (17)
2. If $l'_j(\beta_j, \mathbf{y}) \leqslant 0$ then $x_j(\mathbf{y}) = \beta_j$ (18)
3. If $l'_j(\alpha_j, \mathbf{y}) < 0$ and $l'_j(\beta_j, \mathbf{y}) > 0$ then $x_j(\mathbf{y})$ is the unique solution of the equation $l'_j(x_j, \mathbf{y}) = 0$.

It is easy to verify that this unique solution is given by

$$x_j(\mathbf{y}) = \frac{(p_{0j} + \mathbf{y}^T\mathbf{p}_j)^{1/2} L_j + (q_{0j} + \mathbf{y}^T\mathbf{q}_j)^{1/2} U_j}{(p_{0j} + \mathbf{y}^T\mathbf{p}_j)^{1/2} + (q_{0j} + \mathbf{y}^T\mathbf{q}_j)^{1/2}} \tag{19}$$

Now, since there is an explicit expression for $x_j(\mathbf{y})$, the minimizing x_j in (14), there is also an explicit expression for the dual objective function $W(\mathbf{y})$, namely

$$W(\mathbf{y}) = r_0 - \mathbf{y}^T\mathbf{b} + \sum_{j=1}^{n} \left(\frac{p_{0j} + \mathbf{y}^T\mathbf{p}_j}{U_j - x_j(\mathbf{y})} + \frac{q_{0j} + \mathbf{y}^T\mathbf{q}_j}{x_j(\mathbf{y}) - L_j} \right) \tag{20}$$

Furthermore, the derivatives of $W(\mathbf{y})$, with respect to the dual variables y_i, are given by:

$$\frac{\partial W}{\partial y_i} = -b_i + \sum_{j=1}^{n} \left(\frac{p_{ij}}{U_j - x_j(\mathbf{y})} + \frac{q_{ij}}{x_j(\mathbf{y}) - L_j} \right) \tag{21}$$

The dual problem corresponding to $P^{(k)}$ is the problem of maximizing $W(\mathbf{y})$ over the set of all $\mathbf{y}$ such that $\mathbf{y} \geqslant 0$ (i.e. $y_i \geqslant 0$ for all i).

D: maximize

$$W(\mathbf{y}) \quad \text{subject to} \quad \mathbf{y} \geqslant 0$$

Since $x_i(\mathbf{y})$ depends continuously on $\mathbf{y}$ (except in the rare case that $p_{0j} + \mathbf{y}^T\mathbf{p}_j = q_{0j} + \mathbf{y}^T\mathbf{q}_j = 0$), it follows from (21) that $W(\mathbf{y})$ is a 'smooth' function. It is also easy to prove that $W(\mathbf{y})$ is a concave function (since it is the pointwise minimum of a collection of functions which are linear in $\mathbf{y}$).

D is therefore a rather 'nice' problem, which may be solved by an arbitrary gradient method. We have developed a Fortran subroutine based on a conjugate gradient method, the Fletcher–Reeves method, slightly modified to take care of the non-negativity constraints on the (dual) variables. Apparently, this routine easily solves dual problems containing several hundreds of (dual) variables.

Once the dual problem has been solved, the optimal solution of the (primal) subproblem $P^{(k)}$ is directly obtained by just plugging in the optimal dual solution $\mathbf{y}$ in the expressions for $x_j(\mathbf{y})$ above.

5. ARTIFICIAL VARIABLES

It may happen, in particular during the first iterations if the starting point $\mathbf{x}^{(0)}$ is badly chosen, that a subproblem $P^{(k)}$ becomes infeasible, i.e. without any feasible solutions. In that case, one would still like to obtain a reasonable next iteration point $\mathbf{x}^{(k+1)}$, typically a point which is (in some sense) 'as close to feasible as possible' to $P^{(k)}$.

In order to accomplish this, it is suggested that each subproblem $P^{(k)}$ is modified by the introduction of 'artificial variables' $z_i, i = 1, \ldots, m$, so that the subproblem instead looks as follows (using similar notation as in section 4):

$\tilde{P}^{(k)}$: minimize

$$\sum_{j=1}^{n} \left(\frac{p_{0j}}{U_j - x_j} + \frac{q_{0j}}{x_j - L_j} \right) + \sum_{i=1}^{m} (d_i z_i + d_i z_i^2) + r_0$$

subject to

$$\sum_{j=1}^{n}\left(\frac{p_{ij}}{U_j-x_j}+\frac{q_{ij}}{x_j-L_j}\right)-z_i\leqslant b_i,\quad \text{for}\quad i=1,\ldots,m$$

and

$$\alpha_j\leqslant x_j\leqslant\beta_j,\quad \text{for}\quad j=1,\ldots,n\quad \text{and}\quad z_i\geqslant 0,\quad \text{for}\quad i=1,\ldots,m$$

where $p_{ij}\geqslant 0$, $q_{ij}\geqslant 0$, $L_j<\alpha_j\leqslant\beta_j<U_j$, and $d_i>0$. Each d_i should be a 'relatively large' fixed real number. Obviously, there are always feasible solutions of this problem. (For any $\mathbf{x}$ it is possible to choose $\mathbf{z}$ such that the constraints become satisfied.)

It is easy to prove that if the coefficients d_i are sufficiently large, then all the artificial variables z_i will automatically become zero in the optimal solution of $\tilde{P}^{(k)}$, provided that the unmodified subproblem $P^{(k)}$ (of section 4) is feasible.

On the other hand, if the unmodified subproblem $P^{(k)}$ is infeasible, then some of the z_i will be strictly positive in the optimal solution of $\tilde{P}^{(k)}$. However, because of the high 'cost' of these variables, they will not be greater than absolutely necessary. Therefore, the corresponding $\mathbf{x}$-solution is in some sense as close as possible to being feasible to the unmodified problem $P^{(k)}$.

It should be noted that the dual method described in section 4 may still be used, after some trivial modifications, to solve this new subproblem $\tilde{P}^{(k)}$. In fact, since the number of dual variables y_i is still equal to m, the dual problem of $\tilde{P}^{(k)}$ is not (significantly) harder to solve than the dual problem of $P^{(k)}$.

Concerning the question of how to choose values for the coefficients d_i, it is theoretically sufficient to choose them 'very large' compared to the other coefficients in the objective function. In practice, however, one should probably not choose them unnecessarily large (e.g. 10^{20}), since this in some cases might cause numerical difficulties. It is rather easy, however, to calculate rough estimates of reasonable values on each d_i. Then one might let d_i be equal to such an estimate multiplied by, say, 10 or 100.

6. NUMERICAL TEST RESULTS

MMA (the method of moving asymptotes) has ben coded in Fortran 77 and tested on different problems.

In this section we will report on some (spectacular) results obtained on three different test problems. Even if these problems are purely 'academic', we believe that the obtained results give some insight into the nature of MMA, in particular they indicate a strong potentiality of the method.

(The method has also been implemented at the Aircraft Division of SAAB-SCANIA and tested on some large scale problems containing thickness variables, geometric variables, angular orientation variables, stress constraints, displacement constraints and eigenfrequency constraints. The obtained results have been very satisfactory, but we are not yet ready to present the details concerning these tests.)

Test problem 1: cantilever beam

Consider a cantilever beam, built from 5 beam elements as shown in Figure 1. Each beam element has a quadratic cross-section as shown in Figure 1.

The beam is rigidly supported at node 1, and there is a given external vertical force acting at node 6 (see Figure 1).

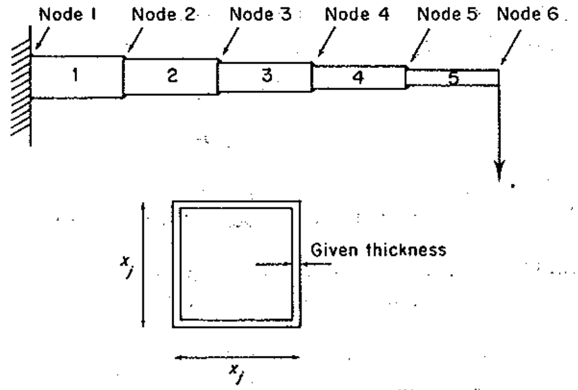

Figure 1. Cantilever beam (test problem 1)

The design variables are the heights x_j of the different beam elements, and the thicknesses are held fixed.

The objective function, to be minimized, is the weight of the beam.

There is only one behaviour constraint, namely a given limit on the vertical displacement of node 6 (where the given load is acting).

The lower bounds on the design variables are so small, and the upper bounds are so large, that they never become active in this problem.

Using classical beam theory, this problem may in fact be stated analytically as follows (after some cleaning):

P1: minimize

$$C_1(x_1 + x_2 + x_3 + x_4 + x_5), \quad x_j > 0$$

subject to

$$61/x_1^3 + 37/x_2^3 + 19/x_3^3 + 7/x_4^3 + 1/x_5^3 \leqslant C_2$$

where C_1 and C_2 are constants whose values depend on material properties, the magnitude of the given load, etc. In our case, it turned out that $C_1 = 0{\cdot}0624$ and $C_2 = 1{\cdot}0$.

P1 may easily be solved analytically. If $C_2 = 1{\cdot}0$, the optimal solution is

$$x_1 = 6{\cdot}016, x_2 = 5{\cdot}309, x_3 = 4{\cdot}494, x_4 = 3{\cdot}502, x_5 = 2{\cdot}153 \tag{22}$$

If, in addition, $C_1 = 0{\cdot}0624$, the corresponding optimal objective value is 1·340.

As a starting point in our tests, we used the feasible solution $x_j^{(0)} = 5{\cdot}0$ for all j. Then the displacement constraint becomes satisfied with equality, and the total weight of this solution is 1·560.

In order to illuminate how the moving asymptotes influence the behaviour of the method, we used the following simple rule for choosing values of $L_j^{(k)}$ and $U_j^{(k)}$:

$$L_j^{(k)} = t x_j^{(k)}, \quad U_j^{(k)} = x_j^{(k)}/t \tag{23}$$

where the parameter t must be chosen strictly between 0 and 1.

Several runs, with different values for the parameter t, have been performed. In particular, the following values were compared:

$$t = 1/16, t = 1/8, t = 1/4, t = 1/3, t = 1/2, t = 2/3, t = 3/4.$$

A 'traditional' method was also included in the tests, namely the method of Reference 2 which, on this problem, coincides with the method of Reference 1. In fact, this method exactly corresponds to the limit case $t=0$ in (23) above. ($L_j^{(k)}=0$ and $U_j^{(k)}\to\infty$)

The move limits $\alpha_j^{(k)}$ and $\beta_j^{(k)}$ were, in all runs, chosen according to the following simple rule:

$$\begin{aligned}\alpha_j^{(k)} &= \max\{0{\cdot}5x_j^{(k)},\quad 1{\cdot}01\,L_j^{(k)}\}\\ \beta_j^{(k)} &= \min\{2{\cdot}0x_j^{(k)},\quad 0{\cdot}99\,U_j^{(k)}\}\end{aligned}\tag{24}$$

For the 'traditional' method this means $\alpha_j^{(k)}=0{\cdot}5x_j^{(k)}$ and $\beta_j^{(k)}=2{\cdot}0x_j^{(k)}$.

The results of the different runs (with different values for t), are shown in Table I. Each iteration

Table I. Results for test problem 1, 'cantilever beam'. The upper entry of each pair is the weight and the second is the infeasibility

Iteration number	'Traditional' method ($t=0$)	MMA with $t=1/16$	MMA with $t=1/8$	MMA with $t=1/4$	MMA with $t=1/3$	MMA with $t=1/2$	MMA with $t=2/3$	MMA with $t=3/4$
0	1·560 0·000	1·560 0·000	1·560 0·000	1·560 0·000	1·560 0·000	1·560 0·000	1·560 0·000	1·560 0·000
1	1·265 0·40	1·274 0·35	1·285 0·23	1·309 0·10	1·327 0·05	1·387 0·000	1·448 0·000	1·477 0·000
2	1·251 0·43	1·270 0·27	1·307 0·11	1·335 0·01	1·338 0·004	1·346 0·000	1·386 0·000	1·418 0·000
3	1·259 0·43	1·304 0·14	1·331 0·03	1·340 0·0005	1·340 0·0001	1·341 0·000	1·358 0·000	1·383 0·000
4	1·250 0·44	1·319 0·08	1·337 0·008				1·347 0·000	1·363 0·000
5	1·258 0·43	1·329 0·04	1·339 0·002				1·343 0·000	1·352 0·000
6	1·249 0·44	1·333 0·02	1·340 0·001				1·341 0·000	1·346 0·000
7	1·258 0·43	1·336 0·01						1·343 0·000
8	· ·	· ·						1·342 0·000
9	· ·	· ·						1·341 0·000
·	·	·						
11	1·259 0·42	1·340 0·002						
12	1·250 0·44	1·340 0·001						
13	1·259 0·42							
· · ·	· · ·							

point $\mathbf{x}^{(k)}$ is represented by two numbers: its objective value (weight) $f_0(\mathbf{x}^{(k)})$ and its 'infeasibility', defined as follows:

$$\text{Infeas}(\mathbf{x}^{(k)}) = \max\{0, \max_i [(f_i(\mathbf{x}^{(k)}) - \hat{f}_i)/\hat{f}_i]\} \tag{25}$$

with notations as in section 2. Clearly, $\mathbf{x}^{(k)}$ is a feasible solution of the original problem P (defined in section 2) if and only if Infeas $(\mathbf{x}^{(k)}) = 0$.

The termination criterion used for this problem is the following: the process is terminated, and the current iteration point $\mathbf{x}^{(k)}$ is accepted as a sufficiently close to optimal solution, as soon as infeas $(\mathbf{x}^{(k)}) < 0{\cdot}001$ and $f(\mathbf{x}^{(k)}) < 1{\cdot}001 f^*$, where f^* is the (known) minimal weight, in our case 1·340.

The obtained results, shown in Table I, are somewhat remarkable. The 'traditional' method did not converge at all (!) whereas for all tested values of $t(1/16 \leqslant t \leqslant 3/4)$ MMA converged to the optimal solution given by (22).

What happened to the 'traditional' method was that, after some iterations, the process oscillated between mainly two different solutions, both being highly infeasible and non-optimal, so that $\mathbf{x}^{(5)} \approx \mathbf{x}^{(7)} \approx \mathbf{x}^{(9)} \ldots$ and $\mathbf{x}^{(4)} \approx \mathbf{x}^{(6)} \approx \mathbf{x}^{(8)} \ldots$ etc. This behaviour was efficiently stabilized by using MMA with a strictly positive t in (23).

The 'best' values of t (for this specific problem!) were those between 1/4 and 1/2. With these values of t the convergence was remarkably fast, only 3 iterations were needed. When $t > 1/2$ the method became somewhat 'too conservative', whereas the opposite was the case when $t < 1/4$. However, even for $t = 1/8$ and $t = 2/3$ only 6 iterations were needed.

Test problem 2:8-bar truss

Next, consider a simple truss structure containing 8 elements (bars) shown in Figure 2.

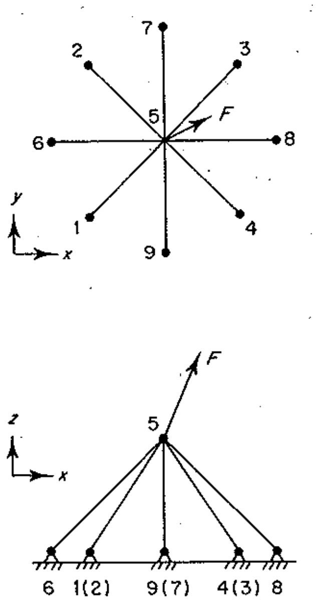

Figure 2. Eight-bar truss (test problem 2)

Topology

Element number	Node numbers	
1	1	5
2	2	5
3	3	5
4	4	5
5	6	5
6	7	5
7	8	5
8	9	5

Geometry

Node number	Node co-ordinates		(mm)
1	−250	−250	0
2	−250	250	0
3	250	250	0
4	250	−250	0
5	0	0	375
6	−375	0	0
7	0	375	0
8	375	0	0
9	0	−375	0

There is only one single load case: an external force $F=(F_x, F_y, F_z)$ at the unsupported node (node 5): $F_x = 40\,\text{kN}, F_y = 20\,\text{kN}, F_z = 200\,\text{kN}$.

The design variables are the cross-sectional areas of the elements. There is no design variable linking. Thus, x_j is the cross-sectional area of the jth bar.

The lower bounds $\underline{x}_j$ on the design variables are $100\,\text{mm}^2$ for all j, and the upper bounds $\bar{x}_j$ are so large that they never become active on this problem.

The objective function, to be minimized, is the weight of the considered structure.

The only behaviour constraints are stress constraints: the stress (tensile or compressive) must not be greater than $100\,\text{N/mm}^2$ in any element under the considered load case.

The following rule is used for choosing values of the asymptotes:

(a) if $k=0$ or $k=1$, then $L_j^{(k)}=0$ and $U_j^{(k)}=5x_j^{(k)}$ for all j.

(b) if $k \geqslant 2$, the rule described at the end of section 3 is applied, i.e. the rule defined by the formulae (12) and (13).

To prevent possible numerical difficulties due to the 'running away' of the asymptotes, it is also required that

$$-50x_j^{(k)} \leqslant L_j^{(k)} \leqslant 0{\cdot}4x_j^{(k)} \quad \text{and} \quad 2{\cdot}5x_j^{(k)} \leqslant U_j^{(k)} \leqslant 50x_j^{(k)}$$

The move limits are simply chosen as follow:

$$\alpha_j^{(k)} = 0{\cdot}5x_j^{(k)} \quad \text{and} \quad \beta_j^{(k)} = 2{\cdot}0x_j^{(k)}$$

As a starting point, we let $x_j^{(0)} = 400\,\text{mm}^2$ for all j.

Several runs have been performed, with different values of the parameter s in the formulae (12) and (13). In our tests, we also included the 'traditional' method of Reference 1 (i.e. linearizing the stress constraints in reciprocal design variables.

It turned out that on this specific problem, the 'traditional' method (corresponding to $L_j^{(k)}=0$) is much too 'conservative', thereby seriously slowing down the convergence.

When MMA was applied it turned out that, after some iterations, $L_j^{(k)}$ and $U_j^{(k)}$ moved away from $x_j^{(k)}$, for all j. Very soon, all the $L_j^{(k)}$ became very negative! It should be noted that the lower the value of the parameter s, the faster could $L_j^{(k)}$ and $U_j^{(k)}$ move away from $x_j^{(k)}$ (see (13)), the more negative become each $L_j^{(k)}$, the larger become each $U_j^{(k)}$, and the less conservative become the approximations.

The results from the different runs are given in Table II. The starting solution $\mathbf{x}^{(0)}$ was infeasible, but after that, all solutions $\mathbf{x}^{(k)}$, in all runs, were feasible. Therefore, each iteration point is, in Table II, represented only by its weight.

The optimal solution, obtained in all the runs, turned out to be (approximately) the following:

$$x_1 = 880\,\text{mm}^2, \quad x_2 = 720\,\text{mm}^2, \quad x_3 = 260\,\text{mm}^2, \quad x_4 = 520\,\text{mm}^2$$
$$x_5 = 100\,\text{mm}^2, \quad x_6 = 100\,\text{mm}^2, \quad x_7 = 100\,\text{mm}^2, \quad x_8 = 100\,\text{mm}^2$$

On this specific problem, it turned out that a non-conservative strategy, compared to the 'traditional' method, is preferable. This problem is therefore in some respect 'the opposite' of the previous problem (the beam), where the 'traditional' method turned out to be much too non-conservative.

Test problem 3: 2-bar truss

Next, consider the simple 2-bar truss in Figure 3, containing one element sizing variable (x_1) and one configuration variable (x_2).

There is one load case: an external force $\mathbf{F}=(F_x, F_y)$ at the unsupported node (node 3), where

Table II. Results (in kg) for test problem 2, '8-bar truss'

Iteration number	'Traditional' method	MMA with $s=3/4$	MMA with $s=1/2$	MMA with $s=1/4$
0	13·05	13·05	13·05	13·05
1	11·68	12·10	12·10	12·10
2	11·66	11·67	11·67	11·67
3	11·64	11·65	11·65	11·65
4	11·62	11·64	11·63	11·61
5	11·60	11·62	11·60	11·52
6	11·59	11·60	11·53	11·42
7	11·57	11·56	11·44	11·28
8	11·55	11·52	11·35	11·23
9	11·53	11·47	11·25	——
10	11·52	11·41	11·23	
11	11·50	11·36	——	
12	11·48	11·31		
13	11·46	11·24		
14	11·45	11·23		
15	11·43	——		
16	11·42			
17	11·40			
18	11·39			
19	11·37			
20	11·36			
·	·			
·	·			
30	11·27			
·	·			
·	·			
39	11·23			

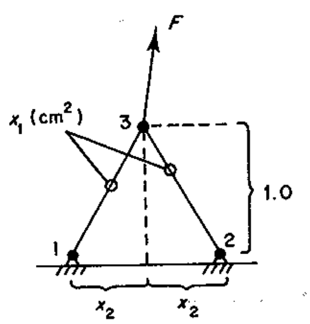

Figure 3. Two-bar truss (test problem 3)

Topology

Element number	Node numbers
1	1 3
2	2 3

Geometry

Node number	Node co-ordinates(m)	
1	$-x_2$	0·0
2	x_2	0·0
3	0·0	1·0

$F_x = 24{\cdot}8$ kN and $F_y = 198{\cdot}4$ kN. ($F_y = 8F_x$ and $|\mathbf{F}| = 200$ kN.)

There are two design variables: x_1, the cross-sectional area (cm^2) of the bars, and x_2, half of the distance (m) between the two nodes 1 and 2.

The lower bounds on the variables are 0·2 cm^2 and 0·1 m, respectively. The upper bounds on the variables are 4·0 cm^2 and 1·6 m, respectively. None of these four bounds becomes active at the optimal solution.

The objective function, to be minimized, is the weight of the bars.

Table III. Results for test problem 3, '2-bar truss'

Iteration number			SLP	Reference 2	MMA	MMA (without move limits)
0	x_1	:	1·50	1·50	1·50	1·50
	x_2	:	0·50	0·50	0·50	0·50
	σ_1	:	0·92	0·92	0·92	0·92
	w	:	1·68	1·68	1·68	1·68
1	x_1	:	1·38	1·39	1·39	1·39
	x_2	:	0·25	0·25	0·25	0·10
	σ_1	:	1·11	1·11	1·10	1·62
	w	:	1·42	1·43	1·43	1·40
2	x_1	:	1·14	1·33	1·22	0·63
	x_2	:	0·50	0·50	0·50	0·62
	σ_1	:	1·22	1·04	1·13	2·23
	w	:	1·27	1·49	1·37	0·74
3	x_1	:	1·34	1·39	1·39	1·45
	x_2	:	0·25	0·25	0·25	0·10
	σ_1	:	1·14	1·11	1·10	1·54
	w	:	1·38	1·43	1·44	1·46
4	x_1	:	1·15	1·33	1·37	1·04
	x_2	:	0·50	0·50	0·38	0·34
	σ_1	:	1·21	1·04	1·03	1·38
	w	:	1·28	1·49	1·47	1·10
5	x_1	:	1·34	1·39	1·41	1·42
	x_2	:	0·25	0·25	0·38	0·40
	σ_1	:	1·14	1·11	1·00	0·99
	w	:	1·38	1·43	1·51	1·53
6	x_1	:	1·15	1·33		1·41
	x_2	:	0·50	0·50		0·38
	σ_1	:	1·21	1·04		1·00
	w	:	1·28	1·49		1·51
7	x_1	:	1·34	1·39		
	x_2	:	0·25	0·25		
	σ_1	:	1·14	1·11		
	w	:	1·38	1·43		
			⋮	⋮		

The (tensile) stress must not be greater than 100 N/mm^2 in either of the two bars, under the considered load case.

Again, this small problem may be formulated analytically (but this is of course not necessary for the method to work!) as follows:

minimize

$$w(x_1, x_2) = C_1 x_1 \sqrt{(1 + x_2^2)}$$

subject to

$$\sigma_1(x_1, x_2) = C_2\sqrt{(1+x_2^2)}\left(\frac{8}{x_1} + \frac{1}{x_1x_2}\right) \leqslant 1 \quad \text{(bar 1)}$$

and

$$\sigma_2(x_1, x_2) = C_2\sqrt{(1+x_2^2)}\left(\frac{8}{x_1} - \frac{1}{x_1x_2}\right) \leqslant 1 \quad \text{(bar 2)}$$

$$0{\cdot}2 \leqslant x_1 \leqslant 4{\cdot}0, \quad 0{\cdot}1 \leqslant x_2 \leqslant 1{\cdot}6$$

where $C_1 = 1{\cdot}0$ and $C_2 = 0{\cdot}124$.

It is obvious from this formulation that the second constraint (i.e. the stress constraint in bar 2) will never become active, since the stress in bar 1 is always strictly greater than the stress in bar 2.

A feasible starting point was chosen, namely $x_1 = 1{\cdot}5\,\text{cm}^2$ and $x_2 = 0{\cdot}5\,\text{m}$, with $w = 1{\cdot}677$ and $\sigma_1 = 0{\cdot}925$.

In MMA, the asymptotes for the sizing variable x_1 were chosen according to the simple formula (23) with $t = 0{\cdot}2$, whereas the asymptotes for the configuration variable x_2 were chosen according to (11), (12) and (13) with $s = 0{\cdot}5$ in (12) and $s = 0{\cdot}75$ in (13).

Two other methods were also tried in the tests: SLP (sequence of linear programmes), which corresponds to $L_j \to -\infty$ and $U_j \to +\infty$, and the method of Reference 2, which corresponds to $L_j = 0$ and $U_j \to +\infty$.

In all the three methods, move limits given by $\alpha_j^{(k)} = x_j^{(k)}/2$ and $\beta_j^{(k)} = 2x_j^{(k)}$ were used.

The results of the tests are shown in Table III. On this problem, neither SLP nor the method of Reference 2 converged. MMA, however, did converge to the optimal solution in about 5 iterations. MMA was also tried without the move limits mentioned above, and it still converged nicely, now in about 6 iterations (the last column in Table III).

7. CONCLUSIONS

We do not claim that the results obtained on the test problems in the previous section are typical in general. On many problems, the method of Reference 2 is known to work very well (i.e. the choice $L_j = 0$ and $U_j = \infty$ works very well).

However, we believe that the obtained results clearly illuminate the importance of the flexibility of MMA, which gives the user some control of the convergence properties of the overall optimization process.

ACKNOWLEDGEMENTS

This work has been financed by the Aircraft Division of SAAB-SCANIA and the Swedish Institute of Applied Mathematics.

REFERENCES

1. L. A. Schmit and B. Farshi, 'Some approximation concepts for structural synthesis', *AIAA Journal*, **12**, 692–699 (1974).
2. C. Fleury and V. Braibant, 'Structural optimization. A new dual method using mixed variables', *LTAS Report SA-115*, University of Liege, 1984.
3. P. Pedersen, 'The integrated approach of FEM-SLP for solving problems of optimal design', in E. J. Haug and J. Cea (eds), *Optimization of Distributed Parameter Structures*, Sijthoff & Noordhoff, 1981.
4. C. Fleury, 'Structural weight optimization by dual methods of convex programming', *Int. j. numer. methods eng.*, **14**, 1761–1783 (1979).
5. K. Svanberg, 'An algorithm for optimum structural design using duality', *Mathematical Programming Study 20*, 1982, pp. 161–177.