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A NEW METHOD FOR OPTIMAL TRUSS TOPOLOGY DESIGN*

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Abstract. Truss topology optimization formulated in terms of displacements and bar volumes results in a large, nonconvex optimization problem. For the case of maximization of stiffness for a prescribed volume, this paper presents a new equivalent, an unconstrained and convex minimization problem in displacements only, where the function to be minimized is the sum of terms, each of which is the maximum of two convex, quadratic functions. Existence of solutions is proved, as is the convergence of a nonsmooth steepest descent-type algorithm for solving the topology optimization problem. The algorithm is computationally attractive and has been tested on a large number of examples, some of which are presented.

Key words. truss topology design, nonsmooth optimization

AMS subject classifications. 90C31, 90C50, 73K40

1. Introduction. Recent years have seen a revived interest in methods for finding optimal topologies of structures [9]. Most work in optimal design of structures is related to optimization of sizes or boundary curves even though it is recognized that optimization of a structural layout (geometry and topology) has an immense impact on the performance of a structure. Analytical methods have been established for the study of fundamental properties of gridlike continua and this field goes back to the work of Michell [12], and is described in monographs by Hemp [8] and Rozvany [17]. Applications of numerical methods to discrete models, especially truss problems, are more recent, with initial studies by Dorn, Gromory, and Greenberg [5]; Fleron [6]; and Pedersen [13]. The last couple of years have seen the development of the so-called homogenization method for generating optimal topologies of structural elements (cf. Bendsøe and Kikuchi [3] and Suzuki and Kikuchi [19]), again emphasizing the great importance of topology design for the performance of a structure.

In this paper, we will consider the problem of finding the stiffest truss which is carrying a given load and which consists of perfect, slender bars of a given total volume. The bars of the truss are a subset of bars connecting all of a number of a priori chosen nodal points, this basic set of bars being the *ground structure* (cf. Fig. 1), and the topology of the truss is generated by varying the cross-sectional areas of the truss, allowing for zero cross-sectional areas. The truss is subject to an external nodal force vector f and the deformation of the truss is described by the vector x of nodal displacements. Figure 2 shows a simple three-bar truss with four nodes, of which three are fixed in all directions. The deformation is thus described by the displacement at the node Z and this displacement is controlled via the equation of equilibrium at this node.

Let a_i, ℓ_i denote the cross-sectional area and length of bar number i, respectively, and assume that all bars are made of the same linear elastic material with Young's modulus E. In order to define equilibrium and to compute bar elongations, construct the *compatibility matrix* $\boldsymbol{B}$, which is a projection matrix that relates nodal forces f and bar forces q by

$$\boldsymbol{B}^T q = f,$$

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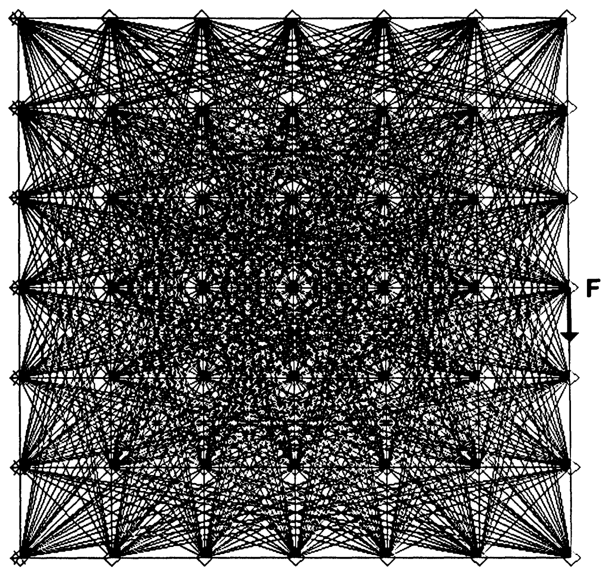

FIG. 1(a). *A ground structure with all possible node connections.*

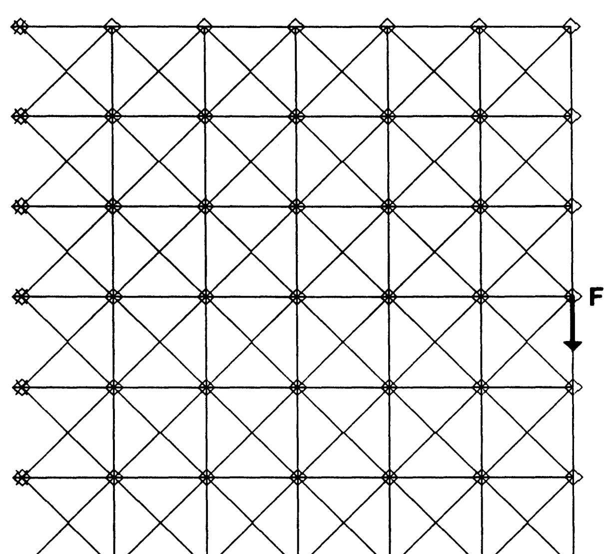

FIG. 1(b). *A ground structure with only neighboring nodes connected.*

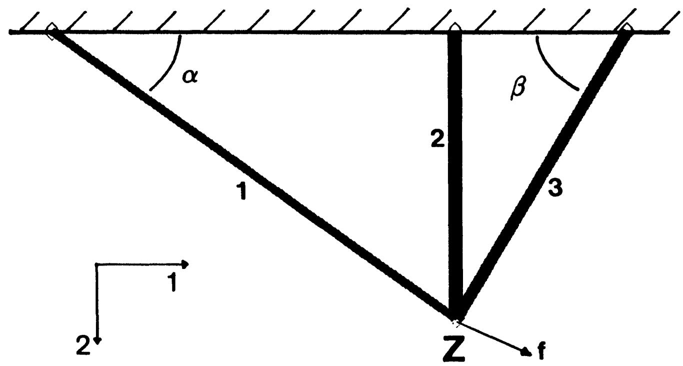

FIG. 2. *A three-bar truss.*

and which relates nodal displacements x and bar elongations Δ by

$$\boldsymbol{B}x = \Delta.$$

For the truss in Fig. 2,

$$\boldsymbol{B} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ 0 & 1 \\ -\cos\beta & \sin\beta \end{pmatrix},$$

as we have three bars and two degrees of freedom. Generally $\boldsymbol{B}$ is an $m \times n_b$ matrix, m being the number of bars and n_b the degrees of freedom; $n_b = (n = \text{no. of nodes}) \times (\text{dim} = \text{dimension of space} = 2 \text{ or } 3) \div (b = \text{no. of support conditions})$.

With a member elongation Δ_i the member force q_i is

$$q_i = \frac{Ea_i}{\ell_i} \cdot \Delta_i, \tag{1.1}$$

so with $\boldsymbol{D} = \text{diag}\,(Ea_i/\ell_i)$, equilibrium is expressed as

$$f = \boldsymbol{B}^T q = \boldsymbol{B}^T \boldsymbol{D}\Delta = \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}x = \boldsymbol{K}x,$$

where $\boldsymbol{K} = \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}$ is called the *stiffness matrix*. The volume of the truss is given as

$$\text{Vol} = \sum_{i=1}^{m} a_i \ell_i,$$

and we thus introduce the volume of each bar, $t_i = a_i\ell_i$, as a more natural variable. Now setting (with $\delta_{k\ell}$ denoting the Kronecker index)

$$(\boldsymbol{D}_i)_{k\ell} = \frac{E}{\ell_i^2}\,\delta_{ik}\,\delta_{k\ell},$$
$$\boldsymbol{K}_i = \boldsymbol{B}^T\boldsymbol{D}_i\boldsymbol{B},$$

the stiffness matrix is written as

$$\boldsymbol{K} = \sum_{i=1}^{m} t_i \boldsymbol{K}_i,$$

where $t_i\boldsymbol{K}_i$ is the element stiffness matrix for element i. For the structure in Fig. 2, the matrices $\boldsymbol{K}_i$ are

$$\boldsymbol{K}_1 = \frac{E}{\ell_1^2}\begin{pmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{pmatrix}, \qquad \boldsymbol{K}_2 = \frac{E}{\ell_2^2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\boldsymbol{K}_3 = \frac{E}{\ell_3^2}\begin{pmatrix} \cos^2\beta & -\cos\beta\sin\beta \\ -\cos\beta\sin\beta & \sin^2\beta \end{pmatrix}.$$

Clearly the matrices $\boldsymbol{K}_i$ are all positive semidefinite. Moreover, it is standard to assume that $\boldsymbol{B}$ has full rank (this depends on the geometry only), so as to exclude rigid body motion or mechanisms. This assumption implies that $\boldsymbol{K}$ is positive definite if all t_i satisfy $t_i > 0$.

The number $f^T x$, called the *compliance* of the structure, is a measure of the work done by the external forces and is thus inversely related to the stiffness of the truss. Finding the stiffest truss for a given total material volume v is thus covered by the formulation

$$(P1) \qquad \min_{x,t} \tfrac{1}{2} f^T x$$

subject to

$$\sum_{i=1}^{m} t_i \boldsymbol{A}_i x = f,$$
$$\sum_{i=1}^{m} t_i = v,$$
$$0 \le L_i \le t_i \le U_i \le \infty,$$

where the design variables t *and* the deformation variables x appear as independent variables, and where $\boldsymbol{A}_i$ are positive semidefinite matrices satisfying the assumption that $\sum t_i \boldsymbol{A}_i$ is positive definite if $t_i > 0$ for all $i = 1, \ldots, m$.

If the truss is supposed to carry a set of different loads, $f^1, \ldots, f^k$, a so-called *multiload problem* can be formulated for the minimization of a weighted average of the compliances for these loads:

$$(Pm) \qquad \min \sum_{p=1}^{k} \tfrac{1}{2} W^p f^p x^p$$

subject to

$$\sum_{i=1}^{m} t_i \boldsymbol{K}_i x^p = f^p, \qquad p = 1, \ldots, k,$$
$$\sum_{i=1}^{m} t_i = v,$$
$$0 \le L_i \le t_i \le U_i \le \infty,$$

where W^p, $p = 1, \ldots, k$, denote suitable weights on the individual compliance values, and x^p are the displacements corresponding to load case f^p. This problem is of a form similar to problem $(P1)$; by introducing an extended displacement vector $x = (x^1, \ldots, x^k)$, an extended, weighted force vector $f = ((W^1)^{1/2}F^1, \ldots, (W^k)^{1/2}f^k)$, and extended unit element stiffness matrices $\boldsymbol{A}_i$ as the block-diagonal matrices with k copies of $\boldsymbol{K}_i$ in the diagonal, problem (Pm) takes the form of problem $(P1)$. In typical applications, the number of loads k is not great, in the order of 2 to 10.

In this paper our main interest is topology design, so we will typically allow for zero cross-sectional areas, i.e., $\ell_i = 0$ for all i. Also, we are primarily seeking to solve problems with a large number of nodal points (e.g., 100) and truss bars, typically taking all connecting bars in the ground structure. With n nodes, we can have up to $m = \frac{1}{2}n(n-1)$ connecting bars, with the total number of variables being $(n_b + m)$ (or $k \cdot n_b + m$ for multiload problems). Thus, for example, a successive quadratic programming (SQP) method typically will not be a suitable method for solving problem $(P1)$ and one should seek to exploit the special structure of the problem, as done in this paper.

The standard approach in structural optimization for a solution procedure for $(P1)$ (see Haftka, Kamat, and Gürdal [7]; and Rozvany and Zhou [18]) is to assume that $L_i > 0$, for all i, so that the state variable x can be eliminated by solving $\boldsymbol{A}x = f$. The derivatives of $f^T x$ are obtained through an adjoint equation, as in optimal control, or through implicit differentiation of the equilibrium equation, and we have

$$\frac{\partial}{\partial t_i}(f^T x) = -x^T \boldsymbol{A}_i x.$$

The problem is then a problem in the design variables t only, but with topology design in mind this is only a very modest reduction in problem size. For many other structural design problems, the number of state variables is much larger than the number of design variables. This is the case in boundary shape optimization with a finite element state model and a boundary defined through a rather small set of spline control points. For such problems, the matrix $\boldsymbol{A}$ is also typically sparse and banded. Again, for topology design, the situation is different because $\boldsymbol{A}$ will typically be neither banded nor sparse, as all nodes are connected.

Note that the topology optimization problem could also be formulated as a discrete optimization problem, but this has mostly been attempted in connection with material selection and cross-section-type selection problems (cf. Kirsch [9]). In addition, the homogenization method developed for topology design of continuum structures has turned out to be capable of generating truss-like thin structures; cf. Suzuki and Kikuchi [19]. This latter method automatically generates the nodal points of the truss and has a discretized formulation analogous to problem $(P1)$, but with $\boldsymbol{A}$ and volume depending nonlinearly on the design parameters. Finally, a natural extension of problem $(P1)$ is to consider the geometric location of the nodal points as design variables as well. These variables would enter the problem through the stiffness matrix $\boldsymbol{A}$ or, rather, through the compatibility matrix $\boldsymbol{B}$. Such a combination has attracted a great deal of attention (see Kirsh [9], Topping [20], and Vanderplaats [21]), but the resulting problem is extremely difficult to solve. With efficient methods for solving high-dimensional problems of type $(P1)$ in its present form, it may be more attractive to introduce a high number of nodal points in the ground structure, and in this way allow for the prediction of the optimal geometric location of nodes.

2. Summary of results. In this paper, we show that the nonconvex optimization problem $(P1)$ can be formulated in terms of an equivalent convex problem in the variables x only, thus achieving a considerable reduction in problem size. The new problem is an unconstrained problem and consists of the minimization of a nondifferentiable function $F(x)$, where $F(x)$ itself is the sum of terms, each of which is the maximum of two convex quadratic functions. For the special case of a problem $(P1)$ with only a zero lower bound on the t_i's (denoted $(P1)_s$), the new formulation is

$$(P2)_s \qquad \min_{x \in R^n} \max_{i=1,\ldots,m} \left\{ \tfrac{v}{2} x^T \boldsymbol{A}_i x - f^T x \right\},$$

where each term $x^T \boldsymbol{A}_i x$ is the *energy* of the bar number i. Note that the optimality conditions for problem $(P1)_s$ are

$$\begin{aligned} x^T \boldsymbol{A}_i x &= \wedge \quad if \;\; t_i > 0, \\ x^T \boldsymbol{A}_i x &\leq \wedge \quad \text{if} \;\; t_i = 0, \\ \sum t_i \boldsymbol{A}_i x &= f, \\ \sum t_i &= v, \qquad t_i \geq 0, \end{aligned} \tag{2.1}$$

where $\wedge$ is the constant (positive) Lagrange multiplier for the volume constraint. We thus see that for the optimal truss topology, no more than $n+1$ active bars (i.e., bars with $t_i > 0$) are needed. (This follows from the optimality conditions (2.1) and Caratheodory Theorem; see, e.g., [14].) Moreover, the active bars all have the same specific energy $x^T \boldsymbol{A}_i x$, and that energy level $\wedge$ is the maximum of the energies in all of the bars. This is reflected in problem $(P2)_s$, as is the fact that the conditions

$$\sum \frac{t_i}{v} \boldsymbol{A}_i x = f/v, \qquad \sum \frac{t_i}{v} = 1 \tag{2.2}$$

imply that a convex combination of the gradients of the energies of active bars equals the load f/v; equation (2.2) thus expresses the fact that the subgradient of the objective function in problem $(P2)_s$ contains zero. As problem $(P1)_s$ is not convex, this equivalence of necessary conditions does not in itself imply equivalence of $(P1)_s$ and $(P2)_s$, but this stronger result is proven in §3, where existence of solutions is also proved. In §4 we present a nonsmooth "steepest descent" algorithm for problem $(P2)$, *which simultaneously solves the original truss topology problem* $(P1)$. Section 5 contains the proof of the convergence of this algorithm. In §6 the algorithm is specialized to problems $(P2)_s$. For this special case, the algorithm is very similar to minmax algorithms, as in Demyanov and Malozemov [4] and Pshenichny and Danilin [14].

Each step of the algorithm consists of a computation of a subset J of bars which for the current estimate of x have a certain fixed energy level. The descent direction can then be computed from a quadratic programming problem with n_b variables and with the number of constraints controlled by J. This QP problem is thus of the same size as the equilibrium equation $\boldsymbol{A}x = f$, but the data of the problem only involves the bars of the set J, which typically contains many fewer than the total number of bars. Alternatively, the dual to this QP can be solved. This dual is also a QP problem, being a least-squares problem in the design variables t_i, $i \in J$, that will generate equilibrium in a least-squares sense for the current estimate of deformation x. It is advantageous to solve the dual problem, as the cardinality of J is usually considerably smaller than n_b. With the descent direction in hand, the steplength of the descent can be computed by an inexact linesearch of the Armijo–Goldstein type. For problem $(P2)_s$ we, in fact, derive an *analytical formula* for the stepsize. Alternatively an exact linesearch (e.g., golden section) can be performed, taking advantage of the fact that in most cases only "almost active" bars will influence the search. In §7 we present a number of computational examples and discuss implementation.

The algorithm is computationally very attractive because the values $x^T \boldsymbol{A}_i x$ require only a few additions and multiplications, and because we avoid assembly of the entire stiffness matrix $\boldsymbol{A} = \sum t_i \boldsymbol{A}_i$ at any stage. The algorithm thus never requires a solution of $\boldsymbol{A}x = f$, and equilibrium is actually first achieved when the algorithm has converged.

For the case of a *single load*, the matrices $\boldsymbol{A}_i$ of problem $(P1)$ are the element stiffness matrices $\boldsymbol{K}_i$, which can be written as

$$\boldsymbol{K}_i = \frac{E}{\ell_i^2} b_i b_i^T, \tag{2.3}$$

where b_i^T is the ith row of the compatibility matrix $\boldsymbol{B}$. In this case, it can be shown (see [1]) that $(P2)_s$ is equivalent to a *linear programming* problem:

$$(LPx) \qquad \begin{aligned} &\min_x -fx \\ &\text{subject to} \\ &1 \le \frac{\sqrt{E}}{\ell_i} b_i^T x \le 1, \quad i = 1, \dots, m, \end{aligned}$$

and this equivalence follows from the nontrivial equivalence between problems $(P1)_s$ and $(P2)_s$. Problem (LPx) has a rather low number of variables, but a very high number of constraints. It should be noted that for multiple load cases and/or upper (and/or lower) bounds on the bar volumes, a similar equivalence to linear programs does *not* hold.

Traditionally, truss topology optimization problems have been formulated in terms of member forces (cf. (1.1)) as a linear programming problem:

$$(LPq) \qquad \begin{aligned} &\min_{q,t} \sum_{i=1}^m t_i \\ &\text{subject to} \\ &\boldsymbol{B}^T q = f, \\ &-t_i \sigma \le \ell_i q_i \le t_i \sigma, \qquad i = 1, \dots, m, \\ &t_i \ge 0 \end{aligned}$$

for minimizing the weight, subject to equilibrium and stress constraints, σ being the limit stress value (see Dorn, Gromory, and Greenberg [5]; Fleron [6]; Kirsch [9], [10]; Pedersen [13]; Ringertz [15]; Topping [20]; and Vanderplaats [21]). *Problem* (LPq) *is the dual of problem* (LPx), written in terms of the variables $q_i = q_i^+ - q_i^-$, $t_i = (q_i^+ + q_i^-)\ell_i/\sigma$, where q_i^+, q_i^- are the dual variables of (LPx).

The equivalence mentioned above (and in other studies [1]) shows that for any solution (t, q) to (LPq) there exists a displacement field x so that (t, x) is a minimum compliance design, i.e., a solution to problem $(P1)$, as it is readily seen that for the member forces $\hat{q}$ corresponding to x, $(t, \hat{q})$ is a solution to problem (LPq). From a design point of view, the variables of primary interest are the bar volumes t_i, so (LPq) is a suitable formulation for plastic as well as elastic design.

3. A displacement-based formulation for truss topology design. The mixed formulation (simultaneous analysis and design) of the truss topology design problem is the following.[1]

PROBLEM $(P1)$.

$$\min_{x\in\Re^n, t\in\Re^m} \tfrac{1}{2}fx$$

subject to

$$\sum_{i=1}^{m} t_i \boldsymbol{A}_i x = f, \tag{3.1}$$

$$\sum_{i=1}^{m} t_i = v, \tag{3.2}$$

$$L_i \le t_i \le U_i, \qquad i = 1, 2, \ldots, m. \tag{3.3}$$

The assumptions on the problem data are

(A1) $0 \le L_i < U_i \le v\,,\quad i = 1, 2, \ldots, m$;
(A2) $\sum_{i=1}^m L_i < v < \sum_{i=1}^m U_i$;
(A3) for every i, the matrix $\boldsymbol{A}_i$ is $n \times n$ symmetric positive semidefinite;
(A4) if $t_i > 0,\ i = 1, \ldots, m$, then the matrix $\sum_{i=1}^m t_i \boldsymbol{A}_i$ is positive definite;
(A5) $f \in \Re^n,\ f \ne 0\,.$

Problem $(P1)$ has a large number of variables $(m + n)$, and is nonconvex in the variables (x, t) due to the constraint (3.1). The main result of this section (Theorem 4) shows that Problem $(P1)$ can be solved by considering an equivalent *convex* programming problem (Problem $(P2)$ below), which has only $n + 1$ variables. Since typically m is much larger than n, Problem $(P2)$ offers an attractive way to solve the truss topology design problem. The formulation of Problem $(P2)$ is as follows.

PROBLEM $(P2)$.

$$\min_{x\in\Re^n, \lambda\in R} \left\{ F(x,\lambda) := \lambda v - fx + \sum_{i=1}^{m} \max\left\{ (\tfrac{1}{2}x\boldsymbol{A}_i x - \lambda)U_i, (\tfrac{1}{2}x\boldsymbol{A}_i x - \lambda)L_i \right\} \right\}.$$

The objective function $F(x, \lambda)$ is a nonsmooth convex function; in fact, it is a piecewise quadratic function, thus of "mild" nonsmoothness. The relation between Problems $(P1)$ and $(P2)$ is given in the following theorem and in Theorem 4 below.

THEOREM 1.

$$\min(P1) = -\min(P2).$$

Proof. Problem $(P1)$ can be written as

$$\min(P1) = \min_t \left\{ g(t) : \sum t_i = v,\ L \le t \le U \right\}, \tag{3.4}$$

where

$$g(t) := \min_x \left\{ \tfrac{1}{2}fx : \sum t_i \boldsymbol{A}_i x = f \right\}. \tag{3.5}$$

[1] To simplify notation, we omit in the sequel the transpose symbol in inner products, matrix multiplications, etc.

We first derive an equivalent expression for $g(t)$. Let $\bar{x} = \bar{x}(t)$ be a solution of (3.5), so that

$$g(t) = \tfrac{1}{2} f\bar{x}(t).$$

Consider the convex problem

$$h(t) := \max_x \{fx - \tfrac{1}{2} x\boldsymbol{A}(t)x\}, \qquad \boldsymbol{A}(t) := \sum t_i \boldsymbol{A}_i. \tag{3.6}$$

The set of optimal solutions of (3.6), $X(t)$, is

$$X(t) = \{x : \boldsymbol{A}(t)x = f\},$$

and since $\bar{x}(t) \in X(t)$,

$$X(t) = \bar{x}(t) + N(\boldsymbol{A}(t))$$

where N denotes "null space." Now $h(t)$ can be computed as

$$\begin{aligned} h(t) &= \max_{x \in N(\boldsymbol{A}(t))} \{f(x + \bar{x}) - \tfrac{1}{2}(x + \bar{x})\boldsymbol{A}(t)(x + \bar{x})\} \\ &= f\bar{x} - \tfrac{1}{2}\bar{x}\boldsymbol{A}(t)\bar{x} + \max_{x \in N(\boldsymbol{A}(t))} \{x(f - \boldsymbol{A}(t)\bar{x}) - \tfrac{1}{2} x\boldsymbol{A}(t)x\} \\ &= \tfrac{1}{2} f\bar{x} \quad \text{since } \boldsymbol{A}(t)\bar{x} = f, \quad \boldsymbol{A}(t)x = 0 \ (x \in N(\boldsymbol{A}(t))). \end{aligned}$$

Thus

$$g(t) = h(t) = \max_x \{fx - z\tfrac{1}{2} x\boldsymbol{A}(t)x\};$$

and substituting this into (3.4),

$$\min(P1) = \min_{\substack{\sum t_i = v \\ L \le t \le U}} \max_{x \in \Re^n} \left\{ fx - \tfrac{1}{2} \sum_{i=1}^{m} t_i (x\boldsymbol{A}_i x) \right\}.$$

This is a minmax problem, which is convex (in fact, linear) in t and concave (quadratic) in x. Moreover, the constraint set of t is compact; hence a minmax theorem (Rockafellar [16, Cor. 37.3.2]) implies

$$\min(P1) = \max_{x \in \Re^n} \min_{L \le t \le U} \left\{ fx - \tfrac{1}{2} \sum_{i=1}^{m} t_i (x\boldsymbol{A}_i x) : \sum_{i=1}^{m} t_i = v \right\}. \tag{3.7}$$

By Lagrange duality, the inner minimization is equal to

$$\begin{aligned} & fx + \max_{\lambda \in R} \min_{L \le t \le U} \left\{ -\tfrac{1}{2} \sum_{i=1}^{m} t_i (x\boldsymbol{A}_i x) + \lambda \left(\sum t_i - v \right) \right\} \\ &= fx + \max_{\lambda \in R} \left\{ - \sum_{i=1}^{m} \max_{L_i \le t_i \le U_i} \{ t_i (\tfrac{1}{2} x\boldsymbol{A}_i x - \lambda) \} - \lambda v \right\} \\ &= fx + \max_{\lambda} \left\{ -\lambda v - \sum_{i=1}^{m} \max\{ (\tfrac{1}{2} x\boldsymbol{A}_i x - \lambda) U_i, \ \tfrac{1}{2}(x\boldsymbol{A}_i x - \lambda) L_i \} \right\}. \end{aligned}$$

Substituting the latter into (3.7),

$$\begin{aligned}\min(P1) &= \max_{x,\lambda}\left\{fx - \lambda v - \sum_{i=1}^{m}\max\left\{\left(\tfrac{1}{2}x\boldsymbol{A}_i x - \lambda\right)U_i,\ \left(\tfrac{1}{2}x\boldsymbol{A}_i x - \lambda\right)L_i\right\}\right\}\\ &= \max_{x,\lambda}\{-F(x,\lambda)\} = -\min(P2). \qquad \square\end{aligned}$$

The next result shows that for Problem $(P2)$ an optimal solution always exists.

THEOREM 2. *There exist $\bar{x} \in \Re^n$, and $\bar{\lambda} \in R$ such that*

$$F(\bar{x},\bar{\lambda}) = \min F(x,\lambda). \tag{3.8}$$

Proof. Let $t^0 \in \Re^m$ be a vector such that

$$t^0 > 0, \quad L \le t^0 \le U, \quad \sum_{i=1}^{m} t_i^0 = v,$$

and let $x^0 \in \Re^n$ be the unique solution of

$$\sum t_i^0 \boldsymbol{A}_i x = f, \tag{3.9}$$

i.e.,

$$x^0 = \left(\sum t_i^0 \boldsymbol{A}_i\right)^{-1} f.$$

(Recall that by assumption (A4), $\sum t_i^0 \boldsymbol{A}_i$ is positive definite and hence nonsingular.)

Let $\lambda^0 \in R$ be fixed but arbitrary. Consider the set

$$S_0 = \{(x,\lambda) \in \Re^n \times R : F(x,\lambda) \le F(x^0,\lambda^0)\}. \tag{3.10}$$

Then

$$\min_{x\in\Re^n,\lambda\in R} F(x,\lambda) = \min_{(x,\lambda)\in S_0} F(x,\lambda).$$

The function $F(x,\lambda)$ is continuous, and so to prove the existence of a solution $(\bar{x},\bar{\lambda})$, it remains to show that S_0 is bounded. Now

$$\begin{aligned}F(x^0,\lambda^0) &= \lambda^0 v - fx^0 + \sum_{i=1}^{m}\max\{(\tfrac{1}{2}x^0\boldsymbol{A}_i x^0 - \lambda^0)U_i,\ (\tfrac{1}{2}x_0\boldsymbol{A}_i x_0 - \lambda_0)L_i\}\\ &\ge \lambda^0 v - fx^0 + \sum t_i^0(\tfrac{1}{2}x^0\boldsymbol{A}_i x^0 - \lambda^0) \quad (\text{since } L_i \le t_i^0 \le U_i)\\ &= \lambda^0\left(v - \sum t_i^0\right) - x^0\left(f - \sum t_i^0 \boldsymbol{A}x^0\right) - \tfrac{1}{2}\sum t_i^0 x^0 \boldsymbol{A}_i x^0\\ &= 0 - 0 - \tfrac{1}{2}fx^0.\end{aligned}$$

So,

$$a_0 := F(x^0,\lambda^0) \ge -\tfrac{1}{2}fx^0. \tag{3.11}$$

Let $(x, \lambda) \in S_0$; then

$$\begin{aligned} a_0 &\geq F(x,\lambda) \geq \lambda v - fx + \sum t_i^0 (\tfrac{1}{2} x \boldsymbol{A}_i x - \lambda) \\ &= -fx + \tfrac{1}{2} x \left(\sum t_i^0 \boldsymbol{A}_i\right) x \\ &\geq -\|f\|\, \|x\| + \tfrac{1}{2}\tau_0\, \|x\|^2, \end{aligned}$$

where $0 < \tau_0$ is the minimum eigenvalue of the (positive definite) matrix $\sum t_i^0 \boldsymbol{A}_i$. The above showed that if $(x, \lambda) \in S_0$ then

$$\tfrac{1}{2}\tau_0 \|x\|^2 - \|f\|\, \|x\| - a_0 \leq 0. \tag{3.12}$$

Consider the polynomial

$$p(\alpha) = \tfrac{1}{2}\tau_0 \cdot \alpha^2 - \|f\|\alpha - a_0.$$

Its discriminant Δ is

$$\Delta := \|f\|^2 + 2\tau_0 a_0 \geq \|f\|^2 - \tau_0 f x^0$$

by (3.11),

$$\geq \|f\|(\|f\| - \tau_0 \|x^0\|)$$

by Cauchy–Schwartz inequality, but

$$\|f\|\, \|x^0\| \geq f x^0 = x^0 \left(\sum t_i^0 \boldsymbol{A}_i\right) x^0 \geq \tau_0 \|x_0\|^2,$$

hence $\|f\| \geq \tau_0 \|x^0\|$, and so, from the above,

$$\Delta \geq 0.$$

Therefore, $p(\alpha)$ has real roots, the larger of which, ρ, is

$$\rho = \frac{1}{\tau_0}\left(\|f\| + \Delta^{1/2}\right) > 0.$$

Now, since $p(\cdot)$ is a convex (quadratic) function ($\tau_0 > 0$), the inequality $p(\alpha) \leq 0$ implies that

$$\alpha \leq \rho < \infty.$$

This shows that (3.12) implies that

$$\|x\| \leq \rho.$$

To derive a bound for λ, whenever $(x, \lambda) \in S_0$, we use the two inequalities

$$\begin{aligned} a_0 &\geq \lambda v - fx + \sum L_i (\tfrac{1}{2} x \boldsymbol{A}_i x - \lambda), \\ a_0 &\geq \lambda v - fx + \sum U_i (\tfrac{1}{2} x \boldsymbol{A}_i x - \lambda). \end{aligned}$$

By assumption (A2), these inequalities imply

$$\frac{-a_0 - fx + \frac{1}{2}\sum U_i x \boldsymbol{A}_i x}{\sum U_i - v} \leq \lambda \leq \frac{a_0 + fx - \frac{1}{2}\sum L_i x \boldsymbol{A}_i x}{v - \sum L_i},$$

which further imply

$$-\frac{a_0 + \|f\|\rho}{\sum U_i - v} \leq \lambda \leq \frac{a_0 + \|f\|\rho}{v - \sum L_i}. \qquad \square$$

Next we derive the necessary and sufficient conditions for $(\bar{x}, \bar{\lambda})$ to be a solution of Problem $(P2)$.

THEOREM 3. *A pair* $(\bar{x}, \bar{\lambda})$, $\bar{x} \in \Re^n$, $\bar{\lambda} \in R$ *is an optimal solution of Problem* $(P2)$ *if and only if there exist multipliers* $\{\bar{t}_i : i = 1, \ldots, m\}$ *such that*

$$\bar{t}_i = L_i \quad \textit{if } i \in J^- := \{j : \tfrac{1}{2}\bar{x}A_j\bar{x} < \bar{\lambda}\}, \tag{3.13}$$

$$\bar{t}_i = U_i \quad \textit{if } i \in J^+ := \{j : \tfrac{1}{2}\bar{x}A_j\bar{x} > \bar{\lambda}\}, \tag{3.14}$$

$$L_i \leq \bar{t}_i \leq U_i \quad \textit{if } i \in J := \{j : \tfrac{1}{2}\bar{x}A_j\bar{x} = \bar{\lambda}\}, \tag{3.15}$$

$$\sum_{i=1}^{m} \bar{t}_i A_i \bar{x} = f, \tag{3.16}$$

$$\sum_{i=1}^{m} \bar{t}_i = v, \tag{3.17}$$

Proof. Since $F(x, \lambda)$ is a convex function, $(\bar{x}, \bar{\lambda})$ solves $(P2)$ if and only if

$$0 \in \partial F(\bar{x}, \bar{\lambda}), \tag{3.18}$$

where ∂F is the subgradient set of F. From well-known results on the subgradient of a sum and of max-functions (see, e.g., Rockafellar [16]), condition (3.18) becomes here

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} -f \\ v \end{pmatrix} + \begin{pmatrix} \sum_{J^-} L_i A_i \bar{x} \\ -\sum_{J^-} L_i \end{pmatrix} + \begin{pmatrix} \sum_{J^+} U_i A_i \bar{x} \\ -\sum_{J^+} U_i \end{pmatrix}$$
$$+ \sum_{J} \operatorname{conv}\left\{ \begin{pmatrix} L_i A_i \bar{x} \\ -L_i \end{pmatrix}, \begin{pmatrix} U_i A_i \bar{x} \\ -U_i \end{pmatrix} \right\}.$$

The latter inclusion holds if and only if numbers $\{\tau_i : i \in J\}$ exist such that

$$\begin{aligned}
&0 \leq \tau_i \leq 1, \qquad i \in J, \\
&f = \sum_{J^-} L_i A_i \bar{x} + \sum_{J^+} U_i A_i \bar{x} + \sum_{J} (\tau_i L_i + (1 - \tau_i) U_i) A_i \bar{x}, \\
&v = \sum_{J^-} L_i + \sum_{J^+} U_i + \sum_{J} (\tau_i L_i + (1 - \tau_i) U_i).
\end{aligned}$$

This system is equivalent to (3.13)–(3.17) with

$$\bar{t}_i = \tau_i L_i + (1 - \tau_i) U_i, \qquad i \in J. \qquad \square$$

The main result follows.

THEOREM 4. *Let $(\bar{x}, \bar{\lambda})$ be an optimal solution of problem $(P2)$, with a corresponding multiplier vector $\bar{t} \in \Re^m$ (see Theorem 3). Then $(\bar{x}, \bar{t})$ is a (global) optimal solution of Problem $(P1)$.*

Proof. Clearly, by (3.13)–(3.17), the pair $(\bar{x}, \bar{t})$ is a feasible solution of $(P1)$. Moreover, by Theorem 1,

$$\begin{aligned}
\min(P1) &= -\min(P2) = -F(\bar{x}, \bar{\lambda}) \\
&= -\bar{\lambda}v + f\bar{x} - \sum_{i=1}^{m} \max\{(\tfrac{1}{2}\bar{x}\boldsymbol{A}_i\bar{x} - \bar{\lambda})U_i,\ (\tfrac{1}{2}\bar{x}\boldsymbol{A}_i\bar{x} - \bar{\lambda})L_i\} \\
&= -\bar{\lambda}v + f\bar{x} - \sum_{J^-}(\tfrac{1}{2}\bar{x}\boldsymbol{A}_i\bar{x} - \bar{\lambda})L_i - \sum_{J^+}(\tfrac{1}{2}\bar{x}\boldsymbol{A}_i\bar{x} - \bar{\lambda})U_i \\
&\quad - \sum_{J} \bar{t}_i(\tfrac{1}{2}\bar{x}\boldsymbol{A}_i\bar{x} - \bar{\lambda}).
\end{aligned}$$

(The last summation is equal to zero by the definition of J.)

$$\begin{aligned}
&= \tfrac{1}{2}f\bar{x} - \bar{\lambda}\left(v - \sum_{J^-} L_i - \sum_{J^+} U_i - \sum_{J} \bar{t}_i\right) \\
&\quad + \tfrac{1}{2}\bar{x}\left(f - \sum_{J^-} L_i\boldsymbol{A}_i\bar{x} - \sum_{J^+} U_i\boldsymbol{A}_i\bar{x} - \sum \bar{t}_i\boldsymbol{A}_i\bar{x}\right) \\
&= \tfrac{1}{2}f\bar{x} - \bar{\lambda}\left(v - \sum_{i=1}^{m} \bar{t}_i\right) + \tfrac{1}{2}\bar{x}\left(f - \sum_{i=1}^{m} \bar{t}_i\boldsymbol{A}_i\bar{x}\right) \qquad \text{(by (3.13), (3.14))} \\
&= \tfrac{1}{2}f\bar{x} \quad \text{(by (3.16), (3.17))}\,.
\end{aligned}$$

So, $(\bar{x}, \bar{t})$ is feasible for $(P1)$ and attains the minimal value: $\min(P1) = \frac{1}{2}f\bar{x}$; hence it is globally optimal. □

The optimality condition for $(\bar{x}, \bar{\lambda})$ to solve $(P2)$ (Theorem 3) reveals that $\bar{\lambda}$ is a *threshold energy level*. All truss members i with energy level $\frac{1}{2}\bar{x}\boldsymbol{A}_i\bar{x}$ below $\bar{\lambda}$ have the minimal volume L_i; all those with energy level above $\bar{\lambda}$ have the maximal volume U_i; all the rest have the *same* energy level $\bar{\lambda}$. We now show how to obtain the threshold value $\lambda = \lambda(x)$ for a given displacement vector x, i.e.,

$$\lambda(x) = \arg\min_{\lambda} F(x, \lambda).$$

The derivation is based on the following lemma.

LEMMA 1. *Let $\bar{v} > 0,\ T_i \geq 0,\ \alpha_i \in R\ (i = 1, 2, \ldots, m)$ be numbers such that*

$$\alpha_1 \leq \alpha_2 < \cdots \leq \alpha_m,$$

$$\sum_{i=1}^{m} T_i > \bar{v}.$$

Let K be the largest integer such that

$$\bar{v} \leq \sum_{i=K}^{m} T_i \qquad (K \leq m).$$

Then, the optimal solution $\bar{\lambda}$ of

$$\min_{\lambda \in R} \left\{ \lambda \bar{v} + \sum_{i=1}^{m} (\alpha_i - \lambda)_+ T_i \right\} \tag{3.19}$$

is $\bar{\lambda} = \alpha_K$.

Proof. Since

$$\sum_{i=K+1}^{m} T_i < \bar{v} \le \sum_{i=K}^{m} T_i,$$

we may write

$$\bar{v} = \theta \sum_{i=K}^{m} T_i + (1-\theta) \sum_{i=K+1}^{m} T_i \quad \text{for some } 0 < \theta \le 1.$$

Now,

$$\begin{aligned}
\min_{\lambda} \{\lambda \bar{v} + \sum_{1}^{m} (\alpha_i - \lambda)_+ T_i\} &= \min_{\lambda} \left\{ \lambda \left(\theta \sum_{K}^{m} T_i + (1-\theta) \sum_{K+1}^{m} T_i \right) \right. \\
&\qquad \left. + \sum_{1}^{m} (\alpha_i - \lambda)_+ T_i \right\} \\
&\ge \theta \min_{\lambda} \left\{ \lambda \sum_{K}^{m} T_i + \sum_{1}^{m} (\alpha_i - \lambda)_+ T_i \right\} \\
&\quad + (1-\theta) \min_{\lambda} \left\{ \lambda \sum_{K+1}^{m} T_i + \sum_{1}^{m} (\alpha_i - \lambda)_+ T_i \right\} \\
&\ge \theta \min_{\lambda} \left\{ \lambda \sum_{K}^{m} T_i + \sum_{K}^{m} (\alpha_i - \lambda) T_i \right\} \\
&\quad + (1-\theta) \min_{\lambda} \left\{ \lambda \sum_{K+1}^{m} T_i + \sum_{K+1}^{m} (\alpha_i - \lambda) T_i \right\} \\
&= \theta \sum_{K}^{m} \alpha_i T_i + (1-\theta) \sum_{K+1}^{m} \alpha_i T_i \\
&= \theta \alpha_K T_K + \sum_{K+1}^{m} \alpha_i T_i := \gamma.
\end{aligned}$$

Substituting $\lambda = \bar{\lambda} = \alpha_K$ in the objective function of (3.19) we get

$$\begin{aligned}
&\alpha_K \left(\theta \sum_{K}^{m} T_i + (1-\theta) \sum_{K+1}^{m} T_i \right) + \sum_{i=1}^{m} (\alpha_i - \alpha_K)_+ T_i \\
&= \theta \alpha_K T_K + \alpha_K \sum_{K+1}^{m} T_i + \sum_{K+1}^{m} (\alpha_i - \alpha_K) T_i \\
&= \theta \alpha_K T_K + \sum_{K+1}^{m} \alpha_i T_i = \gamma,
\end{aligned}$$

so $\bar{\lambda} = \alpha_K$ achieves the lower bound γ. Hence, it is optimal. □

THEOREM 5. *Let $\bar{x} \in \Re^n$ be given, and let*

$$\bar{\lambda} = \arg\min_{\lambda} F(\bar{x}, \lambda).$$

Let $\{i_1, i_2, \ldots, i_m\}$ be a permutation of $\{1, 2, \ldots, m\}$ such that

$$\bar{x} \boldsymbol{A}_{i_1} \bar{x} \leq \bar{x} \boldsymbol{A}_{i_2} \bar{x} \cdots \leq \bar{x} \boldsymbol{A}_{i_m} \bar{x}$$

and let K be the largest integer such that

$$\sum_{j=K}^{m} U_{i_j} + \sum_{j=1}^{K-1} L_{i_j} \geq v \qquad (K \leq m);$$

then

$$\bar{\lambda} = \tfrac{1}{2} \bar{x} \boldsymbol{A}_{i_K} \bar{x}.$$

Proof. $F(x, \lambda)$ can be written as

$$F(x,\lambda) = -fx + \tfrac{1}{2} \sum L_i \, x \boldsymbol{A}_i x + \lambda (v - \sum L_i) + \sum (\tfrac{1}{2} x \, \boldsymbol{A}_i x - \lambda)_+ (U_i - L_i).$$

Hence

$$\bar{\lambda} = \arg\min_{\lambda} \left\{ \lambda (v - \sum L_i) + \sum (\tfrac{1}{2} \bar{x} \boldsymbol{A}_i \bar{x} - \lambda)_+ (U_i - L_i) \right\}.$$

Define

$$\alpha_j := \tfrac{1}{2} \bar{x} \boldsymbol{A}_{i_j} \bar{x}, \qquad j = 1, \ldots, m,$$

$$\bar{v} := v - \sum L_i,$$

$$T_j := U_{i_j} - L_{i_j}, \qquad j = 1, \ldots, m;$$

thus the conclusion in the theorem follows immediately from Lemma 1. □

4. An algorithm for solving Problems $(P2)$ and $(P1)$. We describe an algorithm for solving the nonsmooth problem $(P2)$

$$(P2)\qquad \min_{x\in\Re^n,\lambda\in R}\left\{F(x,\lambda):=\lambda v-fx+\sum_{i=1}^{m}F_i(x,\lambda)\right\},$$
$$F_i(x,\lambda):=\max\{(\tfrac{1}{2}x\boldsymbol{A}_ix-\lambda)U_i,(\tfrac{1}{2}x\boldsymbol{A}_ix-\lambda)L_i\}.$$

The algorithm will find the optimal solution $(\bar{x},\bar{\lambda})$ and will simultaneously generate an optimal solution pair $(\bar{x},\bar{t})$ for Problem $(P1)$. The basic iteration step is

$$\begin{pmatrix}x^{\ell+1}\\ \lambda^{\ell+1}\end{pmatrix}=\begin{pmatrix}x^{\ell}\\ \lambda^{\ell}\end{pmatrix}+\alpha_\ell\begin{pmatrix}d^{\ell}\\ \delta_\ell\end{pmatrix},\qquad \ell=0,1,2,\ldots,$$

where (d^ℓ,δ_ℓ) is a *direction of descent* of F at (x^ℓ,λ^ℓ), and $\alpha_\ell\geq 0$ is the *stepsize*. The direction vector (d^ℓ,δ_ℓ) is generated by solving a quadratic programming problem.

At a given point (x^ℓ,λ^ℓ), the *directional derivative* of F in the direction (d,δ), denoted by $F'(x^\ell,\lambda^\ell,d,\delta)$, is given (using well-known results on the directional derivative of a max-function) by

$$\begin{aligned}F'(x^\ell,\lambda^\ell;d,\delta)&=v\delta-fd+\sum_{J_\ell^-}L_i(d\boldsymbol{A}_ix^\ell-\delta)+\sum_{J_\ell^+}U_i(d\boldsymbol{A}_ix^\ell-\delta)\\&\quad+\sum_{J_\ell}\max\{U_i(d\boldsymbol{A}_ix^\ell-\delta),L_i(d\boldsymbol{A}_ix^\ell-\delta)\},\end{aligned}\tag{4.1}$$

where the index sets $\bar{J}_\ell^-$, $\bar{J}_\ell^+$, and $\bar{J}_\ell$ are defined by

$$\begin{aligned}\bar{J}_\ell^-&:=\{i:\tfrac{1}{2}x^\ell\boldsymbol{A}_ix^\ell<\lambda^\ell\},\\ \bar{J}_\ell^+&:=\{i:\tfrac{1}{2}x^\ell\boldsymbol{A}_ix^\ell>\lambda^\ell\},\\ \bar{J}_\ell&:=\{i:\tfrac{1}{2}x^\ell\boldsymbol{A}_ix^\ell=\lambda^\ell\}.\end{aligned}$$

A *steepest descent direction* of F at (x^ℓ,λ^ℓ) is a vector $(\bar{d}^\ell,\bar{\delta}_\ell)$, which solves the minimization problem

$$\min_{d\in\Re^n,\delta}\{F'(x^\ell,\lambda^\ell;d,\delta)+\tfrac{1}{2}(\|d\|^2+\delta^2)\}.\tag{4.2}$$

The second term in the objective function is added to bound the length of the direction vector (d,δ). Let

$$\begin{aligned}\bar{v}^\ell&:=v-\sum_{\bar{J}_\ell^-}L_i-\sum_{\bar{J}_\ell^+}U_i,\\ \bar{f}^\ell&:=f-\sum_{\bar{J}_\ell^-}L_i\boldsymbol{A}_ix^\ell-\sum_{\bar{J}_\ell^+}U_i\boldsymbol{A}_ix^\ell.\end{aligned}$$

Then, by (4.1),

$$F'(x^\ell,\lambda^\ell,d,\delta)=\bar{v}^\ell\delta-\bar{f}^\ell d+\sum_{\bar{J}_\ell}\mu_i,$$

where

$$\mu_i = \max\{U_i(d\boldsymbol{A}_i x^\ell - \delta), L_i(d\boldsymbol{A}_i x^\ell - \delta)\}.$$

Then, problem (4.2) can be written as a quadratic program in the variables $d \in \Re^n, \delta \in R, \{\mu_i : i \in J_\ell\}$:

$$(P_\ell) \qquad \begin{aligned} &\min\left\{ \bar{v}^\ell\delta - \bar{f}^\ell d + \sum_{\bar{J}_\ell} \mu_i + \tfrac{1}{2}\|d\|^2 + \tfrac{1}{2}\delta^2 \right\}, \\ &\text{subject to} \\ &\mu_i \geq U_i d\boldsymbol{A}_i x^\ell - U_i\delta, \\ &\mu_i \geq L_i d\boldsymbol{A}_i x^\ell - L_i\delta, \qquad i \in \bar{J}_\ell. \end{aligned}$$

One can obtain the optimal solution of (P_ℓ), $(\bar{d}^\ell\bar{\delta}_\ell)$, by solving the dual problem of (P_ℓ), which is as follows (we omit the details):

$$(D_\ell) \qquad \begin{aligned} &\max_t \left\{ -\tfrac{1}{2}\left\| \sum_{\bar{J}_\ell} t_i \boldsymbol{A}_i x^\ell - \bar{f}^\ell \right\|^2 - \frac{1}{2}\left\| \sum_{\bar{J}_\ell} t_i - \bar{v}^\ell \right\|^2 \right\}, \\ &L_i \leq t_i \leq U_i, \qquad i \in \bar{J}_\ell. \end{aligned}$$

From the primal-dual relations between (P_ℓ)–(D_ℓ), if $\bar{t}^\ell$ is the optimal solution of (D_ℓ), then the optimal solution of (P_ℓ) is

$$\bar{d}^\ell = -\left(\sum_{\bar{J}_\ell} \bar{t}_i^\ell \boldsymbol{A}_i x^\ell - \bar{f}^\ell \right),$$

$$\bar{\delta}_\ell = \left(\sum_{\bar{J}_\ell} \bar{t}_i - \bar{v}^\ell \right).$$

It is easy to verify, from the optimality conditions in Theorem 3, and the result of Theorem 4, that the following result holds.

THEOREM 6. $\bar{d}^\ell = 0, \bar{\delta}_\ell = 0$ *if and only if* (x^ℓ, λ^ℓ) *solves Problem* $(P2)$ *and* $(x^\ell, \bar{t}^\ell)$ *solves Problem* $(P1)$.

The last result is of theoretical value since an algorithm based on the iteration step

$$x^{\ell+1} = x^\ell + \alpha_\ell \bar{d}^\ell,$$

$$\lambda^{\ell+1} = \lambda^\ell + \alpha_\ell \bar{\delta}^\ell$$

does not necessarily converge. Indeed, unlike the smooth case, for which the steepest descent algorithm is convergent, this is not the case for nonsmooth problems such as $(P2)$ (see, e.g., Lemarechal [11]).

The cure is to introduce a perturbation of the "active constraint set" J_ℓ. This will prevent the solution of problem (P_ℓ) or (D_ℓ) to change discontinuously when a constraint becomes inactive. The specific way this perturbation is chosen here is described next. In what follows, $\epsilon > 0$ is a fixed parameter controlling the "activity" index sets defined below:

$$J_\ell := \{i : |\tfrac{1}{2}x^\ell \boldsymbol{A}_i x^\ell - \lambda^\ell| \le \epsilon/(U_i - L_i)\},$$

$$J_\ell^+ := \{i : \tfrac{1}{2}x^\ell \boldsymbol{A}_i x^\ell - \lambda^\ell > \epsilon/(U_i - L_i)\},$$

$$J_\ell^- := \{i : \tfrac{1}{2}x^\ell \boldsymbol{A}_i x^\ell - \lambda^\ell < -\epsilon/(U_i - L_i)\}.$$

Also let

$$v^\ell := v - \sum_{i \in J_\ell^+} U_i - \sum_{i \in J_\ell^-} L_i,$$

$$f^\ell := f - \sum_{i \in J_\ell^+} U_i \boldsymbol{A}_i x^\ell - \sum_{i \in J_\ell^-} L_i \boldsymbol{A}_i x^\ell.$$

An *ϵ-steepest descent direction* (d^ℓ, δ_ℓ) for $(P2)$ at (x^ℓ, λ^ℓ) is the solution of the quadratic program $(\hat{P}_\ell)$:

$$(\hat{P}_\ell) \qquad \begin{aligned} &\min_{d,\mu,\delta} \left\{ v^\ell \delta - d f^\ell + \sum_{J_\ell} \mu_i + \frac{1}{2}\|d\|^2 + \frac{1}{2}\delta^2 \right\}, \\ &\text{subject to} \\ &U_i(d\boldsymbol{A}_i x^\ell - \delta + p_i^\ell) - \mu_i \le 0, \qquad i \in J_\ell, \\ &L_i(d\boldsymbol{A}_i x^\ell - \delta + p_i^\ell) - \mu_i \le 0, \end{aligned}$$

where

$$p_i^\ell = \tfrac{1}{2}x^\ell \boldsymbol{A}_i x^\ell - \lambda^\ell.$$

Note that problem $(\hat{P}_\ell)$ is a perturbation of problem (P_ℓ). Indeed $|p_i^\ell| \le \epsilon/(U_i - L_i)$ for $i \in J_\ell$, and $J_\ell \approx \bar{J}_\ell$ for ϵ small; the problems coincide for $\epsilon = 0$.

A dual problem of $(\hat{P}_\ell)$ is the following quadratic program:

$$(\hat{D}_\ell) \qquad \begin{aligned} &\max_t \left\{ \sum_{i \in J_\ell} t_i p_i^\ell - \tfrac{1}{2} \left\| \sum_{i \in J_\ell} t_i \boldsymbol{A}_i x^\ell - f^\ell \right\|^2 - \tfrac{1}{2} \left\| \sum_{i \in J_\ell} t_i - v^\ell \right\|^2 \right\}, \\ &L_i \le t_i \le U_i, \qquad i \in J_\ell. \end{aligned}$$

If t^ℓ is the solution of $(\hat{D}_\ell)$, then the solution (d^ℓ, δ_ℓ) of $(\hat{P}_\ell)$ is given by

$$\begin{aligned} d^\ell &= -\left(\sum_{i \in J_\ell} t_i^\ell \boldsymbol{A}_i x^\ell - f^\ell \right), \\ \delta_\ell &= \left(\sum_{i \in J_\ell} t_i^\ell - v^\ell \right). \end{aligned} \tag{4.3}$$

We now demonstrate that a result similar to Theorem 6 holds for problems $(\hat{P}_\ell)$ and$(\hat{D}_\ell)$.

THEOREM 7. *$d_\ell = 0, \delta_\ell = 0$ if and only if (x^ℓ, λ^ℓ) solves Problem $(P2)$ and (x^ℓ, t^ℓ) solves Problem $(P1)$.*

Proof. The optimality conditions for (d^ℓ, δ_ℓ) to solve problem $(\hat{P}_\ell)$ are

$$\sum_{J_\ell} t_i^\ell - \delta_\ell = v^\ell, \tag{4.4}$$

$$\sum_{J_\ell} t_i^\ell \boldsymbol{A}_i x^\ell + d^\ell = f^\ell, \tag{4.5}$$

$$(t_i^\ell - L_i)\left(U_i(h_i^\ell + p_i^\ell) - \mu_i^\ell\right) = 0, \qquad i \in J_\ell, \tag{4.6}$$

$$(U_i - t_i^\ell)\left(L_i(h_i^\ell + p_i^\ell) - \mu_i^\ell\right) = 0, \qquad i \in J_\ell, \tag{4.7}$$

$$L_i \le t_i^\ell \le U_i, \qquad i \in J_\ell, \tag{4.8}$$

$$\mu_i^\ell = \max\left\{U_i(h_i^\ell + p_i^\ell),\, L_i(h_i^\ell + p_i^\ell)\right\}, \qquad i \in J_\ell, \tag{4.9}$$

where

$$p_i^\ell := \tfrac{1}{2} x^\ell \boldsymbol{A}_i x^\ell - \lambda^\ell, \qquad h_i^\ell := d^\ell \boldsymbol{A}_i x^\ell - \delta_\ell.$$

Define

$$t_i^\ell = L_i, \quad \mu_i^\ell = L_i(h_i^\ell + p_i^\ell), \quad i \in J_\ell^-,$$
$$t_i^\ell = U_i, \quad \mu_i^\ell = U_i(h_i^\ell + p_i^\ell), \quad i \in J_\ell^+.$$

Then, using the definition of v^ℓ and f^ℓ, the system (4.4)–(4.9) can be written as follows:

$$\sum_{i=1}^{m} t_i^\ell - \delta^\ell = v, \tag{4.10}$$

$$\sum_{i=1}^{m} t_i^\ell \boldsymbol{A}_i x^\ell + d^\ell = f, \tag{4.11}$$

$$(t_i^\ell - L_i)\left(U_i(h_i^\ell + p_i^\ell) - \mu_i^\ell\right) = 0, \qquad i = 1, \dots, m, \tag{4.12}$$

$$(U_i - t_i^\ell)\left(L_i(h_i^\ell + p_i^\ell) - \mu_i^\ell\right) = 0, \qquad i = 1, \dots, m, \tag{4.13}$$

$$\begin{gathered} L_i \le t_i \le U_i \qquad i = 1, \dots, m \\ \text{with} \\ t_i^\ell = L_i, \quad i \in J_\ell^-, \quad t_i^\ell = U_i, \quad i \in J_\ell^+, \end{gathered} \tag{4.14}$$

$$\mu_i^\ell = \max\{U_i(h_i^\ell + p_i^\ell),\, L_i(h_i^\ell + p_i^\ell)\}, \qquad i \in J_\ell,$$

$$\mu_i^\ell = L_i(h_i^\ell + p_i^\ell), \qquad i \in J_\ell^-, \tag{4.15}$$

$$\mu_i^\ell = U_i(h_i^\ell + p_i^\ell), \qquad i \in J_\ell^+.$$

The optimality condition at $(\bar{x}, \bar{\lambda})$ for Problem $(P2)$ can be written as the system

$$\sum_{i=1}^m \bar{t}_i = v, \tag{4.16}$$

$$\sum_{i=1}^m \bar{t}_i \boldsymbol{A}_i \bar{x} = f, \tag{4.17}$$

$$(\bar{t}_i - L_i)[U_i \bar{p}_i - \bar{z}_i] = 0, \qquad i = 1, \ldots, m, \tag{4.18}$$

$$(U_i - \bar{t}_i)[L_i \bar{p}_i - \bar{z}_i] = 0, \qquad i = 1, \ldots, m, \tag{4.19}$$

$$L_i \le \bar{t}_i \le U_i, \qquad i = 1, \ldots, m, \tag{4.20}$$

$$\bar{z}_i = \max\{U_i \bar{p}_i, L_i \bar{p}_i\}, \qquad i = 1, \ldots, m, \tag{4.21}$$

where

$$\bar{p}_i := \tfrac{1}{2} \bar{x} \boldsymbol{A}_i \bar{x} - \bar{\lambda}.$$

Let $d^\ell = 0$, $\delta_\ell = 0$. Then $h_i^\ell = 0$. Also $i \in J_\ell^- \Rightarrow p_i^\ell < 0$ and $i \in J_\ell^+ \Rightarrow p_i^\ell > 0$ and therefore (4.15) reduces to

$$\mu_i^\ell = \max\{U_i p_i^\ell, L_i p_i^\ell\}, \qquad i = 1, \ldots, m.$$

It is easily seen, by comparing the systems (4.10)–(4.15) with (4.16)–(4.21), that

$$\bar{x} = x^\ell, \quad \bar{\lambda} = \lambda^\ell, \quad \bar{z}_i = \mu_i^\ell, \quad i = 1, \ldots, m$$

is an optimal solution of $(P2)$.

Conversely, let $\bar{x} = x^\ell, \bar{\lambda} = \lambda^\ell$ be a solution of $(P2)$. Then $p_i^\ell = \bar{p}_i$ and it follows from (4.21) that

$$\bar{z}_i = U_i p_i^\ell \quad \text{if } p_i^\ell > 0, \quad \text{in particular, if } i \in J_\ell^+;$$

$$\bar{z}_i = L_i p_i^\ell \quad \text{if } p_i^\ell < 0, \quad \text{in particular, if } i \in J_\ell^-.$$

Hence, $d^\ell = 0, \delta_\ell = 0$ (which makes $h_i^\ell = 0$) with corresponding multipliers $t_i^\ell = \bar{t}_i$, $\mu_i^\ell = \bar{z}_i$ satisfy the optimality condition (4.10)–(4.15) for $(\hat{P}_\ell)$. □

Once an ϵ-steepest descent direction (d^ℓ, δ_ℓ) has been computed, the stepsize α_ℓ can be computed by

$$\alpha_\ell = \arg\min_{\alpha \ge 0} F(x^\ell + \alpha d^\ell,\ \lambda^\ell + \alpha \delta_\ell).$$

Here, we employ an *inexact linesearch* of the Armijo–Goldstein type. The stopping rule for the algorithm is based on Theorem 7.

ALGORITHM A [For solving Problems $(P2)$ and $(P1)$]
Parameters: $\epsilon > 0$ ("activity" parameter), $\delta > 0$ (for the stopping rule), $0 < \theta < \frac{1}{2}$ (for the stepsize rule).
Initialization

(0.1) *Choose* an initial design vector t^0

$$t^0 > 0, \quad L \le t^0 \le U, \quad \sum_{i=1}^m t_i^0 = v.$$

(0.2) *Solve* the linear system

$$\sum_{i=1}^m t_i^0 \boldsymbol{A}_i x = f$$

to obtain its (unique) solution x^0.

(0.3) *Compute* λ^0 as follows (see Theorem 5). Compute a permutation $(i_1, i_2, \dots, i_m)$ of $\{1, 2, \dots, m\}$ such that

$$x^0 \boldsymbol{A}_{i_1} x^0 \le x^0 \boldsymbol{A}_{i_2} x^0 \le \cdots \le x^0 \boldsymbol{A}_{i_m} x^0.$$

Let K be the largest integer such that

$$\sum_{j=K}^m U_{i_j} + \sum_{j=1}^{K-1} L_{i_j} \ge v \qquad (K \le m).$$

Then

$$\lambda_0 = \tfrac{1}{2} x^0 \boldsymbol{A}_{i_K} x^0.$$

Step ℓ (x^ℓ, λ^ℓ given)

(ℓ.1) Generate the index sets J_ℓ, J_ℓ^+, J_ℓ^-, compute v^ℓ and f^ℓ.

(ℓ.2) Solve $(\hat{P}_\ell)$ to obtain (d^ℓ, δ_ℓ) [OR: solve $(\hat{D}_\ell)$ to obtain t^ℓ and then compute d^ℓ, δ_ℓ by the formula (4.3)].

(ℓ.3) If $\max(\|d^\ell\|, |\delta_\ell|) < \delta$ stop, *else* go to (ℓ.4).

(ℓ.4) Compute the stepsize α_ℓ as the largest $\alpha > 0$ such that

$$F(x^\ell + \alpha d^\ell,\ \lambda^\ell + \alpha\delta_\ell) \le F(x^\ell,\ \lambda^\ell) - \alpha\theta(\|d^\ell\|^2 + \delta_\ell^2). \tag{4.22}$$

Note: An approximation of α_ℓ can be computed as follows. Let $K(\ell) =$ smallest integer K such that $\alpha = (\frac{1}{2})^K$ satisfies (4.22), then

$$\alpha_\ell = (\tfrac{1}{2})^{K(\ell)}.$$

(ℓ.5)

$$x^{\ell+1} = x^\ell + \alpha_\ell d^\ell,$$

$$\lambda^{\ell+1} = \lambda^\ell + \alpha_\ell \delta_\ell.$$

(ℓ.6) $\ell \leftarrow \ell + 1$, go to (ℓ.1).

5. Convergence of Algorithm A. In this section we show that convergent subsequences generated by Algorithm A produce an optimal solution of $(P2)$ and simultaneously (by Theorem 3), an optimal solution of the original truss topology design Problem $(P1)$.

THEOREM 8. *The sequence* $\{x^\ell, \lambda^\ell\}_0^\infty$, *generated by Algorithm* A, *has a convergent subsequence. The limit point of any such subsequence is an optimal solution of problem* $(P2)$.

Proof. In the proof of Theorem 3 it was shown that the set

$$S_0 = \{(x, \lambda) \,:\, F(x, \lambda) \le F(x^0, \lambda^0)\}$$

is compact. Since, by (4.22),

$$F(x^{\ell+1}, \lambda^{\ell+1}) \le F(x^\ell, \lambda^\ell) \qquad \text{for all } \ell = 0, 1, 2, \dots,$$

it follows that

$$\{x^\ell, \lambda^\ell\}_0^\infty \subset S_0,$$

and by the compactness of S_0, this implies the existence of a convergent subsequence. For simplicity of notation, we denote this subsequence also by $\{x^\ell, \lambda^\ell\}_0^\infty$. Let $(\bar{x}, \bar{\lambda})$ be its limit point. Consider an index $i \in J_\ell$; then

$$\begin{aligned} F_i(x^\ell + \alpha d^\ell, \lambda^\ell + \alpha\delta_\ell) &= \max \left\{ \begin{array}{l} U_i(p_i^\ell + \alpha h_i^\ell + \frac{1}{2}\alpha^2 d^\ell \boldsymbol{A}_i d^\ell)\,, \\ L_i(p_i^\ell + \alpha h_i^\ell + \frac{1}{2}\alpha^2 d^\ell \boldsymbol{A}_i d^\ell) \end{array} \right\} \\ &\le \max \left\{ \begin{array}{l} (1-\alpha) U_i p_i^\ell + \alpha\mu_i^\ell + \frac{U_i}{2}\alpha^2 d^\ell \boldsymbol{A}_i d^\ell\,, \\ (1-\alpha) L_i p_i^\ell + \alpha\mu_i^\ell + \frac{L_i}{2}\alpha^2 d^\ell \boldsymbol{A}_i d^\ell \end{array} \right\} \text{ by (4.9)} \\ &\le (1-\alpha) F_i(x^\ell, \lambda^\ell) + \alpha\mu_i^\ell + \tfrac{\alpha^2}{2} M_1 \|d^\ell\|^2 \quad \text{for } 0 < \alpha \le 1, \end{aligned}$$

where

$$M_1 = \max_{i=1,\dots,m} \{U_i \lambda_{\max}(\boldsymbol{A}_i)\}, \qquad \lambda_{\max}(\boldsymbol{A}_i) := \text{maximal eigenvalue of } \boldsymbol{A}_i,$$

and with $p_i^\ell, \mu_i^\ell, h_i^\ell$ defined as in the proof of Theorem 7 (see (4.9)).

From the above inequality

$$\begin{aligned} A &:= \sum_{i \in J_\ell} F_i(x^\ell + \alpha d^\ell, \lambda^\ell + \alpha\delta^\ell) \\ &\le (1-\alpha) \sum_{J_\ell} F_i(x^\ell, \lambda^\ell) + \alpha \sum_{J_\ell} \mu_i^\ell + \tfrac{1}{2}\alpha^2 M_1 \|d^\ell\|^2 m_A, \end{aligned} \tag{5.1}$$

where

$$m_A := \text{card}\,(J_\ell).$$

Note that $d = 0$, $\delta = 0$, $\mu_i = F_i(x^\ell, \lambda^\ell) (= \max\{U_i p_i^\ell, L_i p_i^\ell\})$ is a feasible solution of $(\hat{P}_\ell)$. Hence

$$v^\ell \delta^\ell - d^\ell f^\ell + \sum_{J_\ell} \mu_i^\ell + \tfrac{1}{2}\|d^\ell\|^2 + \tfrac{1}{2}\delta_\ell^2 \le \sum_{J_\ell} F_i(x^\ell, \lambda^\ell), \tag{5.2}$$

so (5.1) and (5.2) imply

$$A \leq \sum_{J_\ell} F_i(x^\ell, \lambda^\ell) - \alpha v^\ell \delta_\ell + \alpha d^\ell f^\ell + \tfrac{1}{2}\alpha(\alpha M_1 m_A - 1)\|d^\ell\|^2 - \tfrac{1}{2}\alpha\delta_\ell^2. \tag{5.3}$$

We now evaluate

$$B := \sum_{J_\ell^+} F_i(x^\ell + \alpha d^\ell, \lambda^\ell + \alpha\delta_\ell) \quad \text{and} \quad C := \sum_{J_\ell^-} F_i(x^\ell + \alpha d^\ell, \lambda^\ell + \alpha\delta_\ell).$$

First, we obtain a bound on $|h_i^\ell|$. Recall

$$h_i^\ell = d^\ell \boldsymbol{A}_i x^\ell - \delta_\ell.$$

Then, by the Cauchy–Schwartz inequality,

$$|h_i^\ell| \leq \|\boldsymbol{A}_i x^\ell\| \, \|d^\ell\| + |\delta_\ell|; \tag{5.4}$$

since

$$(x^\ell, \lambda^\ell) \subset S_0$$

we conclude that (see the proof of Theorem 2)

$$\|x^\ell\| \leq \rho < \infty,$$

and hence

$$M_2 := \max_i \left\{\|\boldsymbol{A}_i x^\ell\|\right\} \leq \rho \max_i \|\boldsymbol{A}_i\| < \infty.$$

By (5.4), then,

$$|h_i^\ell| \leq M_2\|d^\ell\| + |\delta_\ell|. \tag{5.5}$$

Consider the following implications, valid for all $\eta \in R$.

$$0 < \alpha \leq \frac{\epsilon}{(U_i - L_i)|\eta|} \Rightarrow \begin{cases} -\alpha L_i\eta - \epsilon \leq -\alpha U_i\eta, \\ -\alpha U_i\eta - \epsilon \leq -\alpha L_i\eta. \end{cases}$$

Choose $\eta = h_i^\ell = d^\ell \boldsymbol{A}_i x^\ell - \delta_\ell$ and use the bound (5.5) to obtain, for all $i = 1, \ldots, m$

$$\begin{aligned} &0 < \alpha \leq \alpha_1^\ell := \frac{\epsilon}{\max\limits_{i=1,\ldots,m} (U_i - L_i)[M_2\|d^\ell\| + |\delta_\ell|]} \\ &\Rightarrow \begin{cases} -\alpha L_i h_i^\ell - \epsilon \leq -\alpha U_i h_i^\ell & \text{(a)}, \\ -\alpha U_i h_i^\ell - \epsilon \leq -\alpha L_i h_i^\ell & \text{(b)}. \end{cases} \end{aligned} \tag{5.6}$$

Let $i \in J_\ell^+$, i.e.,

$$U_i p_i^\ell > L_i p_i^\ell + \epsilon, \tag{5.7}$$

and let $0 < \alpha \le \alpha_1^\ell$. Then

$$F_i(x^\ell + \alpha d^\ell, \lambda^\ell + \alpha\delta_\ell) = \max\left\{\begin{array}{l} U_i p_i^\ell + \alpha U_i h_i^\ell + \frac{1}{2}\alpha^2 U_i d^\ell \boldsymbol{A}_i d^\ell\,; \\ \\ L_i p_i^\ell + \alpha L_i h_i^\ell + \frac{1}{2}\alpha^2 L_i d^\ell \boldsymbol{A}_i d^\ell \end{array}\right\}$$

$$\le \tfrac{1}{2}\alpha^2 M_1\|d^\ell\|^2 + \max\{U_i p_i^\ell + \alpha U_i h_i^\ell,\ U_i p_i^\ell - \epsilon + \alpha L_i h_i^\ell\}$$

by definition of M_1, (5.7), and the Cauchy–Schwartz inequality,

$$\le \tfrac{1}{2}\alpha^2 M_1\|d^\ell\|^2 + U_i p_i^\ell + \alpha U_i h_i^\ell,$$

by (5.6b).

Since for $i \in J_\ell^+$, $U_i p_i^\ell = F_i(x^\ell, \lambda^\ell)$ and using the definition of h_i^ℓ, the last inequality yields

$$F_i(x^\ell + \alpha d^\ell, \lambda^\ell + \alpha\delta_\ell) \le \tfrac{1}{2}\alpha^2 M_1\|d^\ell\|^2 + F_i(x^\ell, \lambda^\ell)$$
$$+\alpha U_i d^\ell \boldsymbol{A}_i x^\ell - \alpha U_i \delta_\ell \qquad \text{for all } i \in J_\ell^+.$$

Summing for all $i \in J_\ell^+$ we get

$$B \le \sum_{i\in J_\ell^+} F_i(x^\ell, \lambda^\ell) + \alpha\left(\sum_{J_\ell^+} U_i \boldsymbol{A}_i x^\ell\right) d^\ell - \alpha\left(\sum_{J_\ell^+} U_i\right)\delta_\ell$$
$$+ \tfrac{1}{2}\alpha^2 M_1\|d^\ell\|^2 m_B,$$

where $m_B = \text{card}\,(J_\ell^+)$. Similarly we can obtain

$$C \le \sum_{i\in J_\epsilon^-} F_i(x^\ell, \lambda^\ell) + \alpha\left(\sum_{J_\ell^-} L_i \boldsymbol{A}_i x^\ell\right) d^\ell - \alpha\left(\sum_{J_\ell^-} L_i\right)\delta_\ell$$
$$+ \tfrac{1}{2}M_1\|d^\ell\|^2 m_c,$$

where $m_c = \text{card}\,(J_\ell^-)$. Combining the above inequalities for A, B, C, we get

$$F(x^\ell + \alpha d^\ell,\ \lambda^\ell + \alpha\delta_\ell) = A + B + C + \lambda^\ell v - f x^\ell + \alpha\delta^\ell v - \alpha f d^\ell$$
$$\le \sum_{i=1}^m F_i(x^\ell, \lambda^\ell) + \lambda^\ell v - f x^\ell + \alpha\delta_\ell\left(v - \sum_{J_\ell^+} U_i - \sum_{J_\ell^-} L_i - v^\ell\right)$$
$$-\alpha\left(f - \sum_{J_\ell^+} U_i \boldsymbol{A}_i x^\ell - \sum_{J_\ell^-} L_i \boldsymbol{A}_i x^\ell - f^\ell\right) d^\ell$$
$$+\tfrac{1}{2}\alpha(M_1 m\alpha - 1)\|d^\ell\|^2 - \tfrac{1}{2}\alpha\delta_\ell^2.$$

By the definitions of f^ℓ, v^ℓ the last inequality is

$$F(x^\ell + \alpha d^\ell, \lambda^\ell + \alpha\delta_\ell) \le F(x^\ell, \lambda^\ell) + \tfrac{1}{2}\alpha(M_1 m\alpha - 1)\|d^\ell\|^2 - \tfrac{1}{2}\alpha\delta_\ell^2, \tag{5.8}$$

which holds for all $0 < \alpha \leq 1, \alpha \leq \alpha_1^\ell$.

Let $\alpha_2^\ell := ((1-2\theta)/(M_1 m))$ where $0 < \theta < \frac{1}{2}$. Then for $\alpha \leq \alpha_2^\ell$,

$$\tfrac{1}{2}\alpha(M_1 m\alpha - 1) \leq -\theta\alpha.$$

Therefore, by (5.8), for all $0 < \alpha \leq \bar{\alpha}_\ell, 0 < \theta < \frac{1}{2}$,

$$F(x^\ell + \alpha d^\ell, \lambda^\ell + \alpha\delta_\ell) \leq F(x^\ell, \lambda^\ell) - \theta\alpha\|d^\ell\|^2 - \alpha\theta\delta_\ell^2, \tag{5.9}$$

where

$$\bar{\alpha}_\ell = \min(1, \alpha_1^\ell, \alpha_2^\ell).$$

The stepsize α_ℓ in the algorithm is chosen to be the largest $\alpha > 0$ satisfying (5.9). Hence

$$\alpha_\ell \geq \bar{\alpha}_\ell \tag{5.10}$$

and

$$F(x^{\ell+1}, \lambda^{\ell+1}) = F(x^\ell + \alpha_\ell d^\ell, \lambda^\ell + \alpha_\ell\delta_\ell) \leq F(x^\ell, \lambda^\ell) - \theta\alpha_\ell\left(\|d^\ell\|^2 + \delta_\ell^2\right). \tag{5.11}$$

As $\ell \to \infty$, it follows from (5.11) that

$$\alpha_\ell\left(\|d^\ell\|^2 + \delta_\ell^2\right) \to 0. \tag{5.12}$$

Now, by (4.3) and the facts

$$L_i \leq t_i^\ell \leq U_i,$$

$$\|\boldsymbol{A}_i x^\ell\| \leq M_2,$$

it follows that $\|d^\ell\|$ and $|\delta^\ell|$ are bounded above. Hence (see definition of α_1^ℓ in (5.6)) α_1^ℓ is bounded away from zero, and hence also $\bar{\alpha}_\ell$. It follows from (5.10) and (5.11) that, when $\ell \to \infty, \delta_\ell \to 0$ and $d^\ell \to 0$.

As $\ell \to \infty$, we also have

$$x^\ell \to \bar{x}, \quad \lambda^\ell \to \bar{\lambda}, \quad h_i^\ell \to 0, \quad p_i^\ell \to \bar{p}_i, \quad t_i^\ell \to \bar{t}_i.$$

Also, by (4.15), with $\bar{\mu}_i := \lim_{\ell\to\infty} \mu_i^\ell$

$$\begin{aligned} &\bar{\mu}_i = \max\{U_i\bar{p}_i, L_i\bar{p}_i\} && \text{if } |\bar{p}_i| \leq \tfrac{\epsilon}{U_i - L_i}, \\ &\bar{\mu}_i = L_i\bar{p}_i && \text{if } \bar{p}_i < \tfrac{-\epsilon}{U_i - L_i}, \\ &\bar{\mu}_i = U_i\bar{p}_i && \text{if } \bar{p}_i > \tfrac{\epsilon}{U_i - L_i}. \end{aligned}$$

Hence

$$\bar{\mu}_i = \max\{U_i\bar{p}_i, L_i\bar{p}_i\} \qquad \text{for all } i = 1, \ldots, m.$$

Letting $\ell \to 0$ in (4.10)–(4.15), we see that $\bar{x}, \bar{\lambda}, \bar{z}_i = \bar{\mu}_i, \bar{t}_i$ satisfy the optimality condition (4.16)–(4.21) for Problem $(P2)$. Hence $(\bar{x}, \bar{\lambda})$ is its optimal solution. $\square$

6. Truss topology with free design variables. An important special case of the truss topology problem is where the design variable $\{t_i\}$ is free of the upper and lower bounds constraints $L_i \le t_i \le U_i$, i.e., t_i is only required to be nonnegative. Problem $(P1)$ reduces then to

$$(P1)_s \qquad \min\left\{\tfrac{1}{2}fx : \sum_{i=1}^{m} t_i \boldsymbol{A}_i x = f,\ \sum_{i=1}^{m} t_i = v,\ \ t_i \ge 0\right\}.$$

Note that the volume constraint indirectly imposes an upper bound $t_i \le v$. Hence problem $(P1)_s$ is a special case of $(P1)$ with

$$L_i = 0, \quad U_i = v, \quad i = 1, 2, \dots, m.$$

The equivalent displacement-based problem $(P2)$ is then

$$\min_{x,\lambda}\left\{\lambda v - fx + \sum_{i=1}^{m} \max\left\{\left(\tfrac{1}{2}x\boldsymbol{A}_i x - \lambda\right)v, 0\right\}\right\}. \tag{6.1}$$

From Theorem 5, it follows easily that for any given x, the minimizing λ in (6.1) is

$$\lambda = \max_{i=1,\dots,m}\left\{\tfrac{1}{2}x\boldsymbol{A}_i x\right\}.$$

Substituting this value in (6.1), we see that Problem $(P2)$ reduces to a simple convex minmax problem involving only displacement variables:

$$(P2)_s \qquad \min_{x}\left\{F(x) := \max_{i=1,\dots,m}\left\{\tfrac{v}{2}x\boldsymbol{A}_i x - fx\right\}\right\}.$$

For this problem, an ϵ-steepest descent direction $d^\ell \in \Re^n$ of $F(\cdot)$ at x^ℓ is the solution of the quadratic program.

$$\min\{\mu + \tfrac{1}{2}\|d\|^2\} \tag{6.2}$$

$(\hat{P}_\ell)_s$ subject to

$$d^T(v\boldsymbol{A}_i x^\ell - f) + q_i^\ell - \mu \le 0, \qquad i \in I_\ell,$$

where

$$q_i^\ell := \tfrac{v}{2}x^\ell \boldsymbol{A}_i x^\ell - fx^\ell,$$

$$I_\ell := \{i : q_i^\ell > F(x^\ell) - \epsilon\}.$$

The dual problem of $(\hat{P}_\ell)_s$ is here

$$(\hat{D}_\ell)_s \qquad \max_t\left\{\frac{1}{2}\sum_{i\in I_\ell} t_i x^\ell \boldsymbol{A}_i x^\ell - \frac{1}{2}\left\|\sum_{i\in I_\ell} t_i \boldsymbol{A}_i x^\ell - f\right\|^2\right\}$$

$$\text{subject to} \sum_{i\in I_\ell} t_i = v, \quad t_i \ge 0, \quad i \in I_\ell.$$

ALGORITHM B [For solving $(P2)_s$]
Parameters: $\epsilon > 0$ (activity), $\delta > 0$ (stopping rule), $0 < \theta < \frac{1}{2}$ (stepsize rule),
Initialization: *Choose* $t^0 > 0$, $\sum_{i=1}^m t^0 = v$, *compute* x^0, the unique solution of

$$\sum t_i^0 \boldsymbol{A}x = f.$$

Step ℓ (x^ℓ given)

(ℓ.1) *Compute* q_i^ℓ, $F(x^\ell) = \max_{i=1,\dots,m}\{q_i^\ell\}$ and the index set I_ℓ.

(ℓ.2) *Compute* the search direction d^ℓ by solving the quadratic program $(\hat{P}_\ell)_s$, or by solving the dual $(\hat{D}_\ell)_s$, to obtain the solution t^ℓ, and then set

$$d^\ell = -\left(\sum_{I_\ell} t_i^\ell \boldsymbol{A}_i x^\ell - f\right).$$

(ℓ.3) If $\|d^\ell\| < \delta$ stop, x^ℓ is the solution of $(P2)_s$ [t^ℓ is the solution of $(P1)_s$] *else*, go to (ℓ.4).

(ℓ.4) *Compute* the stepsize α_ℓ by the formula

$$\alpha_\ell = \min_{i=1,\dots,m}\{\alpha_i^\ell\}, \tag{6.3}$$

where

$$\alpha_i^\ell = \begin{cases} -c_i^\ell/b_i^\ell & \text{if } a_i^\ell = 0, \quad b_i^\ell > 0, \\ \dfrac{-b_i^\ell + \sqrt{(b_i^\ell)^2 - 4a_i^\ell c_i^\ell}}{2a_i^\ell} & \text{if } a_i^\ell > 0, \\ \infty & \text{if } a_i^\ell = 0, \quad b_i \le 0; \end{cases}$$

here the numbers a_i^ℓ, b_i^ℓ, c_i^ℓ are given by

$$a_i^\ell = \tfrac{v}{2} d^\ell \boldsymbol{A}_i d^\ell \ge 0,$$

$$b_i^\ell = d^\ell(v\boldsymbol{A}_i x^\ell - f) + \theta\|d^\ell\|^2,$$

$$c_i^\ell = q_i^\ell - F(x^\ell) \le 0;$$

(ℓ.5) $x^{\ell+1} = x^\ell + \alpha_\ell d^\ell$;

(ℓ.6) $\ell \leftarrow \ell + 1$, go to (ℓ.1).

To explain the analytic formula (4.3) for the stepsize α_ℓ, we first note that the stepsize rule (4.22) in Algorithm A reduces in our special case to

$$\alpha_\ell \text{ is the largest } \alpha \ge 0 \text{ such that } \quad F(x^\ell + \alpha d^\ell) \le F(x^\ell) - \alpha\theta\|d^\ell\|^2. \tag{6.4}$$

We now prove the following theorem.

THEOREM 9. *The stepsize given by* (6.3) *is the solution of* (6.4).

Proof. Inequality (6.4) is specifically

$$\tfrac{v}{2}(x^\ell + \alpha d^\ell)\boldsymbol{A}_i(x^\ell + \alpha d^\ell) - f(x^\ell + \alpha d^\ell) \le F(x^\ell) - \alpha\theta\|d^\ell\|^2, \qquad i = 1,\dots,m,$$

which further reduces to

$$\alpha d^\ell(v\boldsymbol{A}_i x^\ell - f) + \tfrac{v}{2}\alpha^2 d^\ell \boldsymbol{A}_i d^\ell + q_i^\ell \le F(x^\ell) - \alpha\theta\|d^\ell\|^2, \qquad i = 1, \dots, m. \tag{6.5}$$

Define, for $i = 1, \dots, m$,

$$\varphi_i(\alpha) := \alpha^2 \left(\tfrac{v}{2} d^\ell \boldsymbol{A}_i d^\ell\right) + \alpha \left[d^\ell(v\boldsymbol{A}_i x^\ell - f) + \theta\|d^\ell\|^2\right] + q_i^\ell - F(x^\ell).$$

Then (6.5) is just

$$\varphi_i(\alpha) \le 0, \qquad i = 1, \dots, m. \tag{6.6}$$

Now,

$$\varphi_i(0) = q_i^\ell - F(x^\ell) \begin{cases} = 0 & \text{if } i \in I^0(x^\ell), \\ < 0 & \text{otherwise;} \end{cases}$$
$$I^0(x^\ell) := \{i : \ q_i^\ell = F(x^\ell)\},$$

and

$$\varphi_i'(0) = d^\ell(v\boldsymbol{A}_i x^\ell - f) + \theta\|d^\ell\|^2.$$

Recall that d^ℓ (together with μ_ℓ) is an optimal solution of $(\hat{P}_\ell)_s$; since $d = 0$, $\lambda = \max\{q_i^\ell\} = F(x^\ell)$ is a feasible solution of $(\hat{P}_\ell)_s$, we have

$$\mu_\ell + \tfrac{1}{2}\|d^\ell\|^2 \le F(x^\ell). \tag{6.7}$$

Therefore, for $i \in I^0(x^\ell)$, it follows from (6.2), (6.7) that

$$d^\ell(v\boldsymbol{A}_i x^\ell - f) + \tfrac{1}{2}\|d^\ell\|^2 \le 0,$$

and since $0 < \theta < \frac{1}{2}$,

$$d^\ell(v\boldsymbol{A}_i x^\ell - f) + \theta\|d^\ell\|^2 < 0,$$

i.e.,

$$\varphi_i'(0) < 0 \quad \text{for } i \in I^0(x^\ell). \tag{6.8}$$

From the above discussion, the stepsize α_ℓ solving (6.4) is given by

$$\alpha_\ell = \arg\max\{\alpha : \ \varphi_i(\alpha) \le 0, \alpha > 0\}.$$

Each function φ_i is convex, and

$$\begin{aligned} &\varphi_i(0) = 0, \quad \varphi_i'(0) < 0, \quad i \in I^0(x^\ell), \\ &\varphi_i(0) < 0, \qquad i \notin I^0(x^\ell). \end{aligned} \tag{6.9}$$

Thus (see Fig. 3), each φ_i has at most one root α_i^ℓ in $(0, \infty)$ and

$$\alpha_\ell = \min_{i=1,\dots,m} \{\alpha_i^\ell\}. \tag{6.10}$$

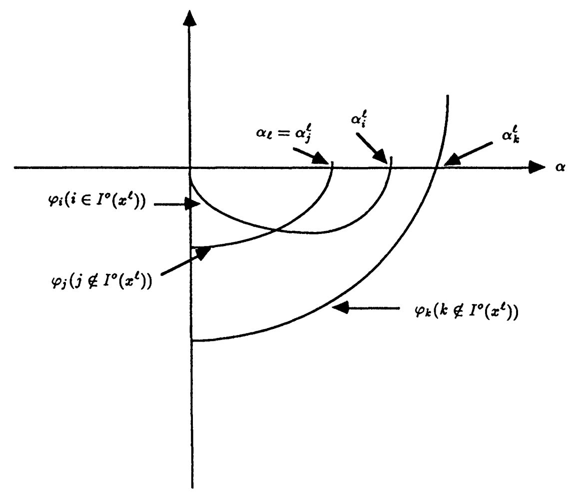

FIG. 3. *Computation of the stepsize.*

Denote the coefficients of the quadratic function $\varphi_i(\cdot)$ by

$$a_i^\ell = \tfrac{v}{2} d^\ell A_i d^\ell \geq 0,$$

$$b_i^\ell = d^\ell(vA_i x^\ell - f) + \theta\|d^\ell\|^2,$$

$$c_i^\ell = q_i^\ell - F(x^\ell) \leq 0.$$

Then α_i^ℓ is given by

$$\alpha_i^\ell = \begin{cases} -c_i^\ell / b_i^\ell & \text{if } a_i^\ell = 0, \quad b_i^\ell > 0, \\ \frac{-b_i^\ell + \sqrt{(b_i^\ell)^2 - 4a_i^\ell c_i^\ell}}{2a_i^\ell} & \text{if } a_i^\ell > 0, \\ \infty & \text{if } a_i = 0, \quad b_i \leq 0, \end{cases}$$

and so (6.10) agrees with (6.3). □

7. Computational results. In this section, we will present a number of results obtained by using Algorithm B. For clarity, we concentrate on $(P1)_s$ with free design variables $(t_i \geq 0)$. Thus, this section will deal with the implementation of Algorithm B.

First, we note that the algorithm only requires computation of vectors $A_i x$ and numbers $y^T A_i x$. Thus we need not assemble nor store the matrices A_i, nor must we assemble the entire matrix A at any iteration step. The compatibility matrix should also not be stored (each column contains at most 2× dim nonzero elements), but instead one works with the $2 \times m$ matrix of connectivities, giving the numbers of the nodal points to which a given bar is connected, as well as a matrix of bar cosines. This means that even though our primal variables are connected to the nodal points, all computations and storage are based on bar numbers. In our implementation, the search vector d was always computed by solving the dual problem (D_ℓ) (or $(\hat{D}_\ell)_s$) in the active bar volumes t_i (i.e., J_ℓ or I_ℓ), as it is our experience that the number of (almost) active bars is considerably less than the number of degrees of freedom for the full truss. Finally, for the linesearch, both the (analytic) Armijo–Goldstein search and an exact linesearch have been tried. It turns out that the inexact search is typically very conservative and that the exact linesearch, especially for larger problems, gives a better performance. In the implementation of the linesearch (golden section method), in order to save costly function-calls, we do not use *all* bars, but only a subset of the $\hat{\epsilon}$-active one ($\hat{\epsilon}$ is larger than ϵ, typically $\hat{\epsilon} = 10\epsilon$). The full set of bars is used only if such a search does not improve the value of the objective function.

For the truss topology optimization (with $L_i = 0$), we are interested in the ultimate set of active bars

$$I_0(x^*) = \{i : \, t_i^* > 0\}.$$

It is true, however (see similar claims in, e.g., [4]), that for all sufficiently small $\epsilon > 0$, there exists a neighborhood N^* of x^* such that

$$I_\epsilon(x) = I_0(x^*) \quad \text{for all } x \in N^*.$$

It is thus natural to work with a decreasing sequence of ϵ-values. It was found that it is important not to choose ϵ too small for the first iterations, and that it is a good strategy to work with a sequence of alternatingly decreasing values of the ϵ-parameter as well as the stopping parameter δ. We note here that the final δ should be at least small enough that we can accept δ as an error in the satisfaction of the equilibrium equations.

The special problem $(P1)_s$ is made up of expressions which are elementwise linear in all variables, except geometric data. Thus, for a specific choice of ground structure geometry and load vector *direction*, the optimal topology only needs to be computed for one set of assigned values of Young's modulus E, volume v, load size f, and geometric scale; for any other values of these variables, the optimal values of the design variables t, the deformation x, and the compliance fx can be derived by a simple scaling. Thus, $(P1)_s$ lends itself to the creation of a "catalogue of optimal topologies" for both single and multiple loads. The optimal compliance may then conveniently be given in terms of the nondimensional compliance ϕ,

$$\phi = (f^T x) v \, E/(\|f\|^2 \ell^2),$$

where ℓ is a typical length dimension (horizontal length of truss in the examples that follow). For Problem $(P1)$, the optimal compliance should also be given in terms of ϕ and the bounds, L_i, U_i, in terms of ratios of the volume v.

Examples of optimal topologies are shown in Figs. 4–7. In these examples, where all connections between nodal points are used as the ground structure, overlapping connecting bars between two nodal points have been removed so as to avoid a redundancy in the model and a trivial possibility of subspaces of optimal solutions. In the optimal topologies, some straight bars appear with intermediate nodal points with no other connecting bars. Such bars should be thought of as straight bars without these intermediate nodal points, as a truss model under the given load will not be able to distinguish between the two configurations.

The final topology and the performance of the optimal structure depend intimately on the choice of ground structure, as does the performance of the algorithm. If the optimal topology consists of only a very low number of bars, the algorithm predicts this very quickly, even though the potential number of bars is large. However, it is also required

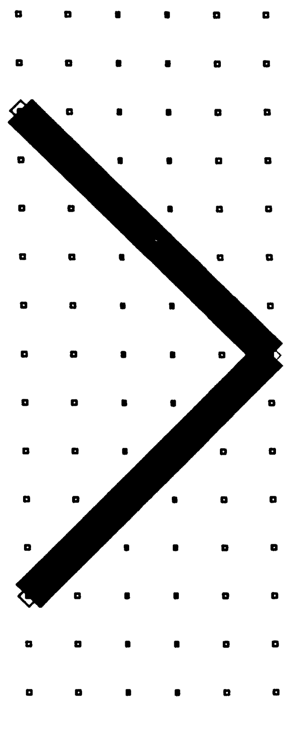

FIG. 4(a). *The optimal truss for a ground structure with* 2852 *potentials bars.*

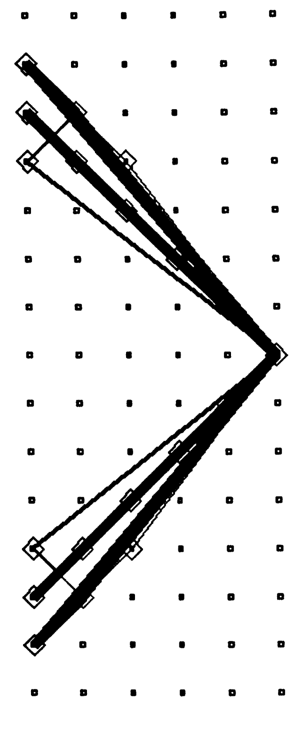

FIG. 4(b). *The optimal truss for the same ground structure as in Fig.* 4(a) *but with upperbounds on bar volumes.*

that "nature's optimal topology" is indeed a subset of the bars in the ground structure; if not, the algorithm will find approximations (however, the topologies are optimal for each choice of ground structure), usually involving many bars. It is well known that the best structure for carrying a single load which is parallel to a line of possible support is a two-bar truss with trusses at 45^o to the line of support (cf. Rosvany [17]). Such a situation is mimicked in all the examples shown, but only the structure in Fig. 4 allows for this optimum as part of its ground structure. The ground structure of Fig. 4 consists of all 2852 nonoverlapping connections between the equally spaced 6×16 nodes in a 10×30 rectangle. All left-hand nodes are possible supports and the single vertical force is at the mid right-hand node. Figure 4(a) shows the optimal, two-bar truss obtained when no constraints on the bar volumes t_i are imposed and the optimal nondimensional compliance is 4.0. In Fig. 4(b), upper bounds on the bar volumes are imposed, as $U_i = 0.01 \cdot \ell_i \cdot v$, and the compliance ϕ increases to 4.1092. The result in Fig. 4(a) was computed using Algorithm B and the result in Fig. 4(b) is the result of using Algorithm A; for the latter example, the deformation field x of Fig. 4(a) was used as the starting point of the algorithm. Notice that introducing upper bounds on the design variables, as expected,

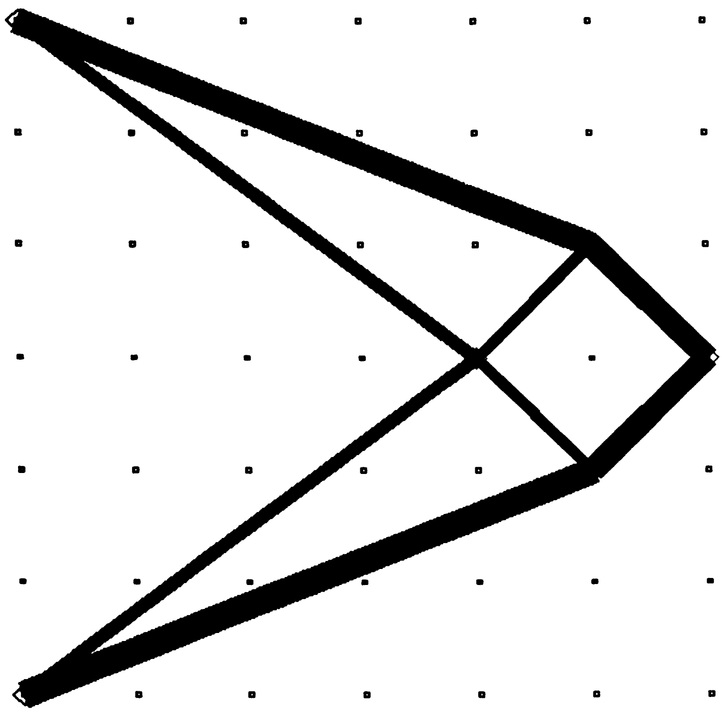

FIG. 5(a). *The optimal truss for the ground structure of Fig.* 1(a).

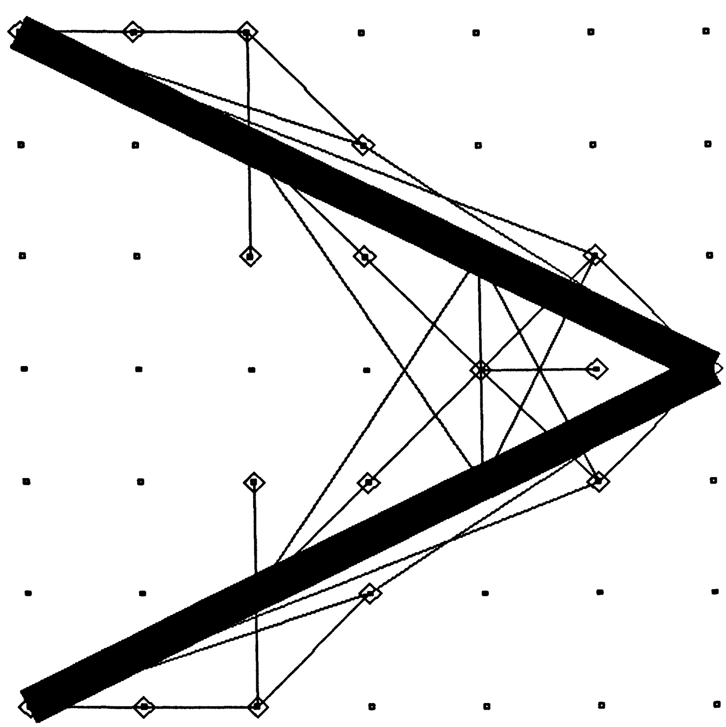

FIG. 5(b). *The optimal multiload design of a truss corresponding to the ground structure of Fig.* 1(a).

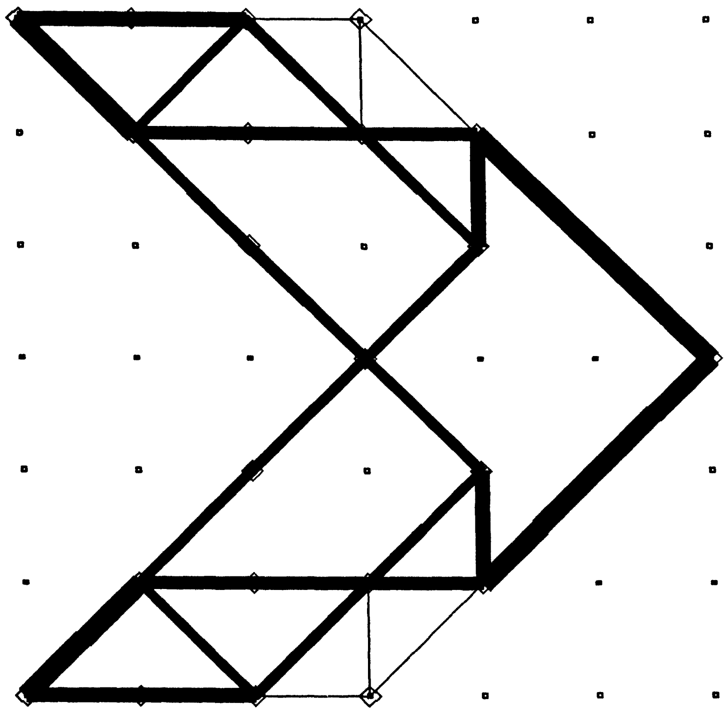

FIG. 6(a). *The optimal truss for the ground structure of Fig.* 1(b)*—single-load case.*

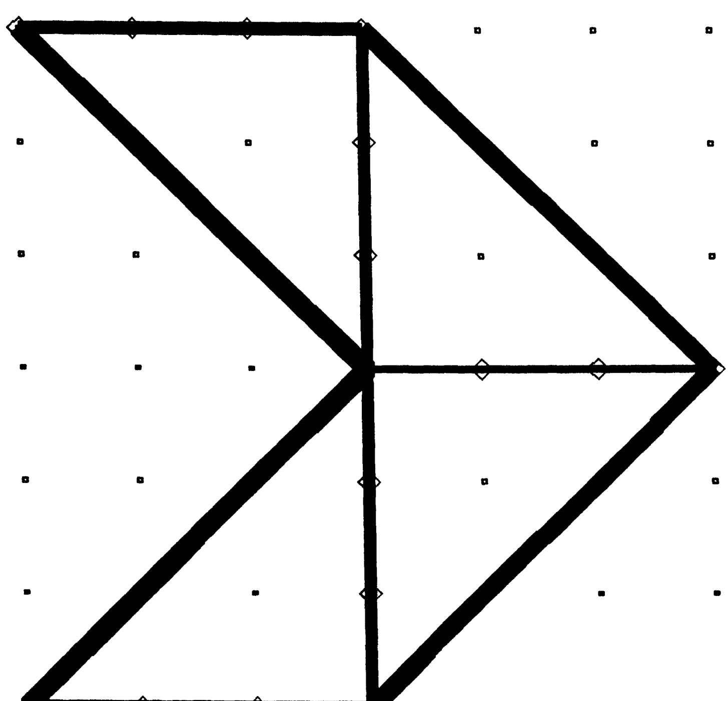

FIG. 6(b). *The optimal truss for the ground structure of Fig.* 1(b)*—three-load cases.*

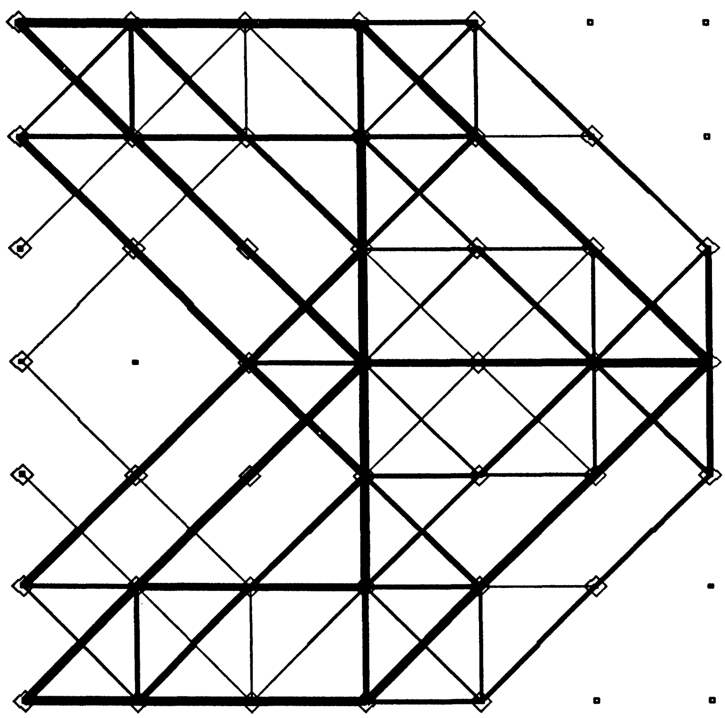

FIG. 6(c). *The optimal truss for the structure of Fig.* 6(b) *but with upper bounds on bar volumes.*

increases the number of bars in the structure as well as increasing the number of bars in the active set J_ℓ.

In Fig. 5(a) we show the optimal, unconstrained truss topology for the ground structure and loading condition of Fig. 1(a). The compliance ϕ is 6.0134, i.e., 1.5 times greater than for the two-bar truss of Fig. 4(a). In Fig. 5(b), an extra, horizontal load has been added at the loaded node and the figure shows the multiload design obtained for unconstrained design variables. The horizontal and vertical loads are equal in size and the weights on the compliances are 1.0 and 2.0, respectively. The average nondimensional compliance is 4.6943 and the compliances for each of the loads are 6.2541 and 1.5747, respectively. The multiload problem results in what is in practice a two-bar truss (trusses at $\pm\ 30^o$ with horizontal direction), thus giving a simpler geometric layout. This feature is even more apparent in the example of Fig. 6, where we use the ground structure of Fig. 1(b). In Figure 6(a), we have the one-load case corresponding to the ground structure in Fig. 1(b), while in Fig. 6(b), we have three load cases: a horizontal and a vertical load at the mid, a right-hand node and a vertical load at the mid node, all of equal size and weighted 1.0, 2.0 and 1.0, respectively. Finally, in Fig. 6(c), we have a design-constrained ($U_i = 0.01 \cdot v \cdot \ell_i$) topology for the same ground structure and set of loads. For the unconstrained problem, the average compliance is 6.3737 and the individual compliances are 3.752, 9.4577, and 2.8273; with constraints the values are 7.2108 and 4.1401, 10.5957, and 3.5117.

In Fig. 7 we illustrate the effect of increasing the number of nodal points (and potential bars) for a ground structure geometry for which "nature's optimal topology" is

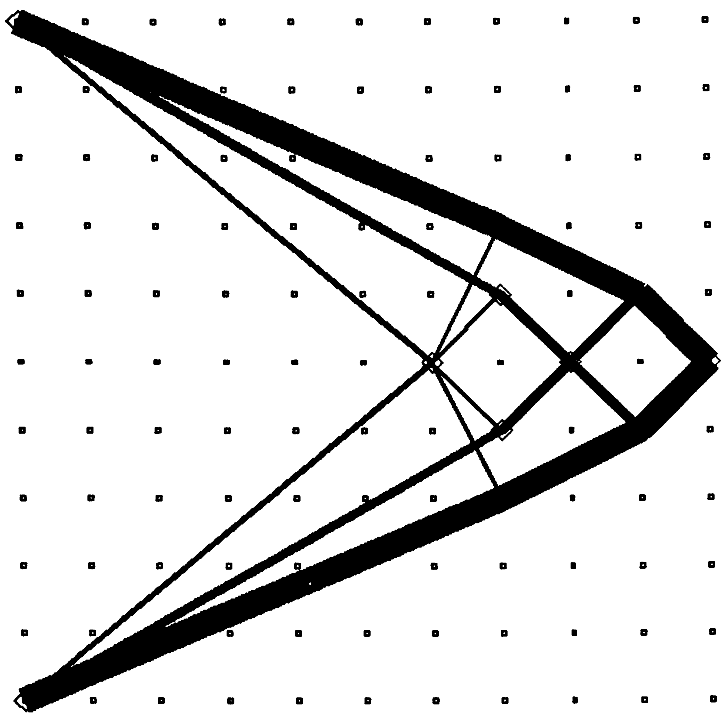

FIG. 7(a). *The optimal truss for the structure with the same geometry and load as in Fig.* 1(a), *with* 11×11 *nodes* (4492 *potential bars*).

a so-called Michell truss [12], [9], i.e., a curve-linear layout of a continuum of unidirectional load-bearing members. In Fig. 7, we have the same geometry and load as in Fig. 1(a). We allow all connections between nodes and have increased the number of nodes to an 11×11 (Fig. 7(a)) and a 15×15 (Fig. 7(b)) equidistant layout of nodes, giving 4492 and 15556 nonoverlapping connections and 5.9646 and 5.9344 nondimensional compliances, respectively. The number of bars in the optimal topology increases dramatically as the layout tries to mimic the curved layout of the optimum Michell truss, thus approximating a layout which is at the limit of the range of a truss model; similar behavior is seen in plate optimization and shape design (cf. [3]). The high number of active bars in the final topology slows the algorithms considerably and indicates that it is important to make a suitable choice of ground structure when optimizing topology.

Finally, it should be noted that the optimal compliance value fx^* is not very sensitive to variations in the values of the design variables. Small variations in the cross-sectional areas of the bars in the optimal topology and even the addition or deletion of thin bars have very little influence on the stiffness of the truss, as measured by compliance. Also, multiple solutions seem to exist, especially in cases with possible symmetry. These remarks are but experimental observations. However, some of them can be substantiated thoeretically by using results from, e.g., [2].

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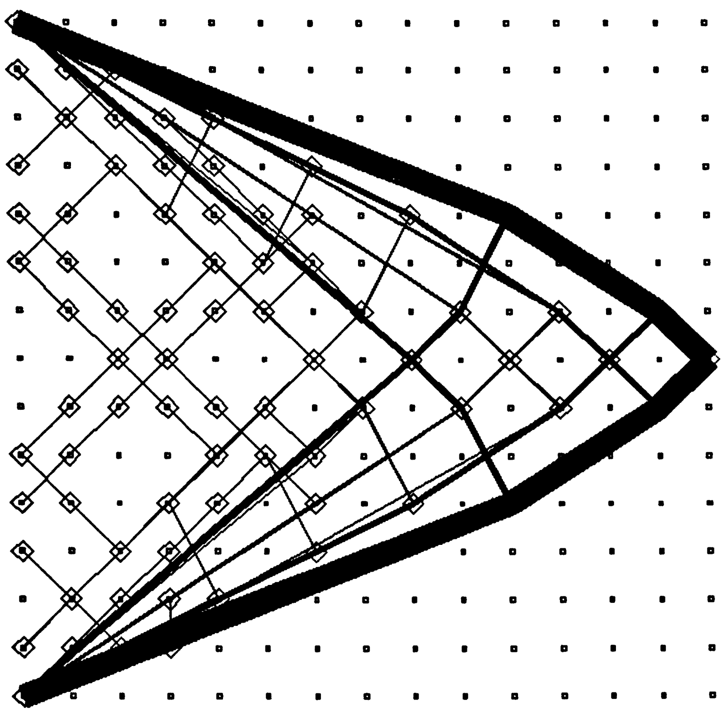

FIG. 7(b). *The optimal truss for the structure with the same geometry and load as in Fig.* 1(a), *with* 15×15 *nodes* (15556 *potential bars*).

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