# Curved Crease Folding
# a Review on Art, Design and Mathematics

**Erik D. DEMAINE**
Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, USA
*edemaine@mit.edu*

**Martin L. DEMAINE**
Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, USA
*mdemaine@mit.edu*

**Duks KOSCHITZ**
Department of Architecture, Massachusetts Institute of Technology, USA
*duks@mit.edu*

**Tomohiro TACHI**
Graduate School of Arts and Sciences, The University of Tokyo, JAPAN
*tachi@idea.c.u-tokyo.ac.jp*

(Authors are listed alphabetically.)

## Summary

In this paper we review masterpieces of curved crease folding, the deployed design methods and geometric studies on this special kind of paper folding. Our goal is to make this work and its techniques accessible to enable further development. By exploring masterpieces of the past and present of this still underexplored field, this paper aims to contribute to the development of novel design methods to achieve shell structures and deployable structures that can accommodate structural properties and design intention for a wide range of materials.

**Keywords:** *origami, curved creases; paper folding; structural morphology; digital fabrication.*

## 1. Introduction

Origami or paper folding is not only a great source of inspiration in architectural design, but is also an effective medium for structural form finding because the developability and foldability characteristics of origami are useful for designing shells and folding/unfolding deployable structures.

Traditional paper folding mostly uses straight creases. We call this type of origami prismatic origami, since straight creases surround planar facets and compose a polyhedral surface (e.g., PCCP shells and Miura-ori [1] and Resch's structure [2]). Here, by altering the crease and making it into a curved folding, the surface suddenly becomes a complex three-dimensional form that cannot be described easily by simple parameters as vertex coordinates. Curved folding is a hybrid of folding and bending a sheet, and the surface is comprised of curved creases and smooth developable surface patches. This can be compared to prismatic origami being the result of pure folding, and the smooth developable surface created from pure bending of a sheet.

The hybrid property of curved folding has an advantage when used to form a 3D surface from sheet materials. When we try to form a surface by pure bending, the shape is limited to simple geometries such as cones, cylinders and tangent surfaces. Prismatic origami on the other hand is more flexible in design, but cannot represent a smoothly curved surface without increasing the resolution by a sufficient number of creases. Such creases form a large number of vertices where the material largely deforms in-plane. Here, a curved fold forms a variety of surfaces using mostly separated small number of creases. Design examples of curved folding used for forming 3D surface are shown in Section 3.

Furthermore, there is a relatively new approach for applying curved folding to structure. A curved folding can be discretized as a planar quadrangle mesh, the interpretation of which as a rigid origami yields a mechanical linkage with 1 degree of freedom. Hence the curved folding can be used for designing transformable structures as shown in [3].

Our goal is to find out further different types of applications of curved folding and make curved folding applicable in a wider context of design by understanding its form variations and the geometry behind it. In this sense, curved folding is a relatively underexplored topic. Therefore, we start from introducing previous works by artists and designers and the geometric approach applied to analyze and design curved foldings. Here are the main contributions of our paper.

1. We introduce the works by a variety of artists and designers who have deeply explored the forms of curved folding.
2. We show examples of curved folding used for the design of products and interior fixtures and the design procedure adopted.
3. We review successful geometric analysis of curved folding and the design methods based on geometric and computational means.

## 2. Curved Creases in Art and Design

### 2.1 Napkin folding

When we consider examples of curved creases, the trajectories of art, mathematics and education will cross and we will observe concurrent developments in all fields. Napkin folding is surprisingly well documented in German since the 17th century, but not as art rather as a teaching document. This decorative art with its complicated table decorations required manuals to teach all techniques. An early account of such a handbook with curved creases is the *Trincir-Buch* by Georg Philipp Harsdoerffer from 1652 [4].

### 2.2 The Bauhaus model

The teaching work by Josef Albers at the first Bauhaus in 1927 and 1928 is documented in photographs and represents the first account of a specific curved crease model, which other origami experts have investigated [7]. This model is made of concentric circles with alternating mountain and valley folds and automatically folds into the shape seen in Figure 1 on the left [8]. Students made this model in Josef Albers' "Vorkurs", an introductory design class. He decided to teach design through paper models, because it is an abstract exercise that allows students to focus only on the design and the paper, not on pragmatic or functional requirements for instance. He promotes to let students try out designs without any a priori knowledge of architecture or established design methodologies. He calls this "non-expert experimentation". The material itself is the only constraint, which is very much aligned with the then newly established Bauhaus tradition. Albers points out that working with materials and exploiting its properties for a design leads to a fundamental understanding of efficiency of means. A material will be utilized to its maximum potential, which

*Fig. 1: The Bauhaus design (left) and a variation design by Erik Demaine and Martin Demaine (right), both models by Martin Demaine*

leads to light design proposals with little waste. Lastly designing curved crease models provides students with unexpected revelations, which was a pedagogical value that Albers appreciated [9].

Another model was recreated by Irene Schawinsky, the wife of Alexander "Xanti" Schawinsky, who was a Bauhaus student and later taught at Black Mountain College during the time Albers was teaching there. Her model shows a variation with a large hole in the center [10]. Thoki Yenn publicized his version of the model in the 80s, which he called "Before the Big Bang" [11]. Kunihiko Kasahara learned of the model from Yenn and made many variations, which he published in *Extreme Origami* in 2003 [12]. Erik and Martin Demaine started to explore this model together in 1989 and made many variations of it since. The model shown in Figure 3 differs from the original as multiple discs of paper are joined together [13]. The sculptures are part of the Museum of Modern Art (MoMA) permanent collection. In 2008 the Demaines and Duks Koschitz created further variations of this model that are based on crease patterns that use conic sections. The resulting shapes display very different symmetries than the symmetries of the flattened crease pattern [14].

### 2.3 Books on paper folding with curved creases

Less known for his paper foldings but certainly recognized for his bookmaking art, Kurt Londenberg (1914-1995) published "Papier und Form" featuring works of paper folding in 1972 with several new editions later on. He presents paper folding in various contexts including a section called "architectural folding". Many of the photographed models were made specifically for the book and he saw this publication as an educational contribution [9]. Londenberg attributes great significance to Bauhaus educator and artist, Josef Albers, and reprinted his article on working with paper. In the same year Hiroshi Ogawa [15] published crease patterns in his *Forms of Paper* and both authors should be considered as designers since they made their geometrically repeating sculptures themselves. Their works display artistic qualities beyond the didactic role they played in their books, but the authors refrained from elaborating artistic motivations.

### 2.4 Two important figures of the 70s

The most expressive work from the 70s that is not related to the Bauhaus model but uses curved creases has to be attributed to computer scientist David Huffman and artist Ron Resch. The contemporaries knew one another and had many discussions about paper folding. Huffman remained true to his roots and took an analytical approach, while Resch was more interested in applied techniques for sculptures and other artistic endeavors. Both published and had a strong connection to computational processes, but only Resch used computers to realize some of his sculptural work [16]. Huffman's work can be described as true to the "one piece of paper, no cuts" rule of folding purists as seen in Figure 2 on the left. He never talked about origami and always referred to his work as paper folding. Resch, being concerned with fabrication methods and the expressive nature of art created sculptures with elaborate boundaries.

Huffman may have not considered himself to be an artist, but his work is highly valued in the folding community both as artistic artifacts and mathematical investigations. He mentioned in a description of himself while teaching at UCSC. *"I don't claim to be an artist. I'm not even sure how to define art. But I find it natural that the elegant mathematical theorems associated with paper surfaces should lead to visual elegance as well."* [17]

*Fig. 2: "column with cusps" by David Huffman. Reconstructed model by Duks Koschitz*

Huffman's passion beyond his academic work was rooted in paper folding. He focused on tessellations with straight creases early on and it is hard to estimate when exactly he discovered curved creases for himself. Huffman owned the book by Hiroshi Ogawa with its curved crease patterns and several examples in this publication are comparable to Huffman's own investigations. It is however unclear when exactly he acquired the book. Ogawa's sculptures are fairly regular and his goal was to cover many techniques [15], but Huffman's investigations are far more rigorous. While we do not think that the craft culture of the 70s had a big impact on the development of curved creases it might still be worthwhile to point out that Thelma, Jay and Lee Newman have collected a great deal of relevant examples in the paper issue of their *The Complete Book of…* series on crafts [18]. Included are two curved-crease examples Huffman investigated himself and it is possible he knew of this very popular book series.

### 2.5 Contemporary art

Contemporary examples of paper foldings in Figure 3 by Robert SweeneyYuko Nishimura and T.Roy Iwaki continue to intrigue art focused audiences and display how regular repeated shapes have been used in art successfully. Sweeney systematically tackles certain configurations and focuses on creating free standing or suspended objects often made of many pieces [19] and in some case at large scales. Matthew Shilan, also an artist, states that he works with what he calls 'systems' and that he does not know what the result will be. Once a system of folding is initiated, the outcome is unknown, led as it is by the qualities of the material. According to him the process consists of 'exploration and invention' [20]. Nishimura on the other hand has a connection to her art through folding day to day commodities from kimonos to wrapping goods in gift shops. "She does not focus on any specific area of origami tessellations and is interested in expressing the Japanese soul

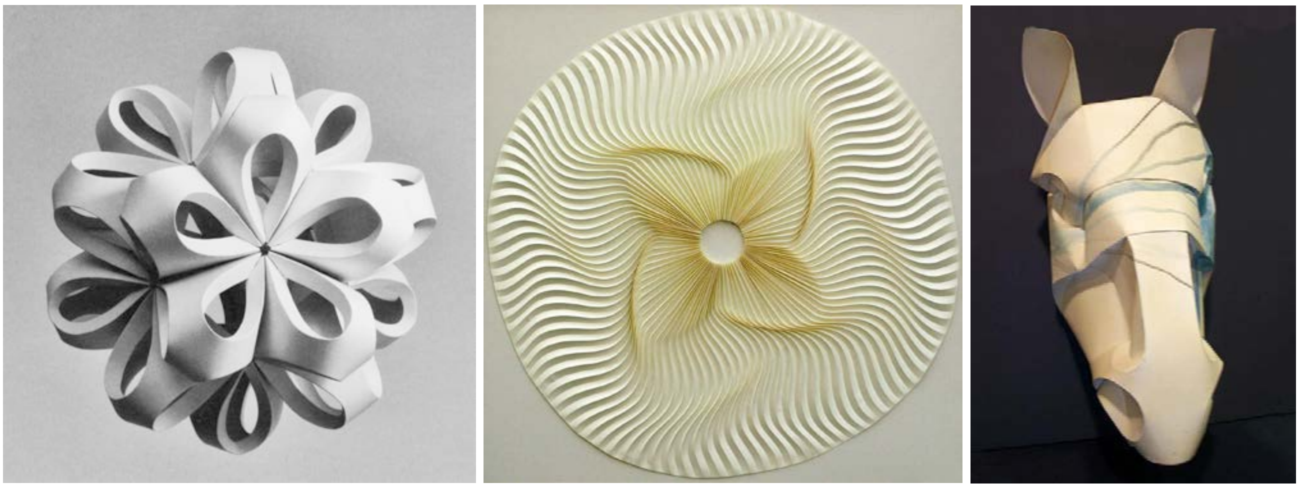

*Fig. 3: Design by Richard Sweeney (photo by the artist) (left), Yuko Nishimura (photo by Yosuke Otomo)(middle) and T.Roy Iwaki (photo by Robert Lang) (right).*

through form" [20].

It may seem to be necessary to discuss Paul Jackson's work in this context, specifically his "one fold models", but when looked at closely it becomes evident that while the paper is curved the creases are straight lines [21]. The resulting sculptures are obviously developable, but do not use curved creases as the previously mentioned examples do. His work consists of very regular but expressive shapes and we will consider designs with less regularity in applied design fields.

Saadya Sternberg studies geometric tessellations and published in the OSME series. Sternberg created a catalog of spiral tessellations and elaborates on techniques how to use curved creases. He also recreated a Huffman model known as "hexagonal column with cusps" [22].

Roy T. Iwaki created elaborate origami masks of animals and based his designs on simple basic shapes that he then used in complex aggregations to achieve the necessary concave and convex portions of an animal's headas the one in Figure 3 on the right [23].

## 3. Design Implementations in Product and Interior Design

Industrial designers, product designers and interior architects sometimes privilege practical approaches over artistic methods and since we know little about this geometry designer's opinions and descriptions of their own work can enlighten us in terms of how to design with something we don't fully understand. The choice of working with a specific material for a design project is in part political, cultural and historical, and while we believe that one needs to discuss design with that in mind, we will focus on formal aspects here. When materiality is intrinsically related to the process we will elaborate on the role of the material in relationship with the formal operations.

### 3.1 Lamp Design by the LeKlint Company

The lamp designs by the LeKlint company are still produced by hand toady. The designs had been developed in the family and it is unclear when exactly the first examples of curved creases were created. The shapes consist of cylindrical configurations that create continuous surfaces. Long narrow plastic sheets are folded into their shape in Figure 4, very much the same way Huffman made his models [5].

*Fig. 4: Lamp by the LeKlint Company*

### 3.2 Bench Design by Tim Herok, Markus Schein

The "Liegengenerator" by Tim Herok and Markus Schein helps generate bench designs. The process starts with defining tight constraints for two edges of the bench design. One edge touches the ground and the other is the center line in the symmetry plane of the seating area [24]. Schein set up a digital model that is using a genetic algorithm [25], which is looking for a solution that a user customizes by tweaking height and undulation parameters. The resulting plan spline represents the outline on the floor and is used to construct the section spline. After intersection points are plotted the surface can be constructed by lofting the discrete parts together. This case shows us that designers are drawn to this kind of geometry and are willing to accept a very constrained base premise in order to realize their project. This design approach however is different from the previous ones as it places the designer further in the background. User defined parameters generate the final shape that was selected by an algorithm.

### 3.3 Metal Column Covers by Haresh Lalvani

Haresh Lalvani's work on column covers [26] is made with a very similar approach that uses a genetic algorithm to exhaustively explore a simple setup. The expressiveness of their "orchestrated random" designs is remarkable when one takes the constraints into consideration. The column covers are part of the MoMA collection and are made in regular and stainless steel.

### 3.4 Car Design by Gregory Epps

Gregory Epps designed the car in Figure 5 as proof of concept prior to founding is company RoboFold. The resulting shapes of his method that starts by crumpling paper are irregular and have great expressive potential. Epp's car does not need to enclose a volume as opposed to the LeKlint designs since the bottom of a vehicle is rarely controlled by the designer. Epp's playful design method is also not conducive to creating such a configuration as this is very difficult to make surfaces meet in a continuous way.

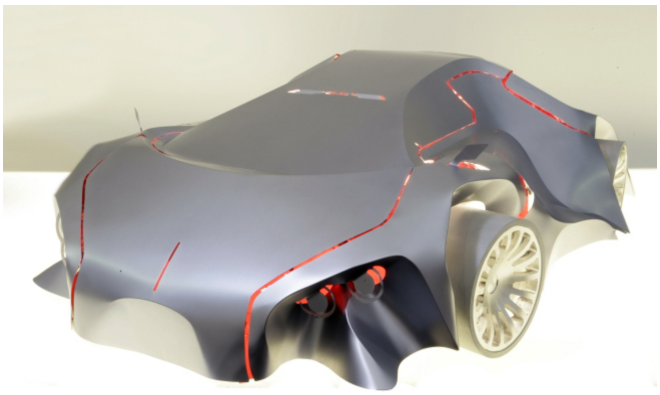

*Fig. 5: Gregory Epps' car design (photo by the artist)*

In concluding the design section we would like to point out a curious characteristic common to all mentioned examples, which is their bias towards symmetry. We believe this is a combined result of functional requirements and the difficulties designers face when working with curved creases, which might change once we know more about this geometry.

# 4. Mathematical Analysis and Computational Methods

In order to morphologically investigate curved foldings to make them flexibly applicable to different design purposes, it is necessary to understand the geometric nature of curved folding. One of the difficult problems of curved folding is that it is defined by continuously smooth surface and non-smooth creases. The surface cannot be represented by simple parameters like NURBS surfaces because of the restrictive constraints induced from its geometry. There is no known universal representation of curved folding. Here, we introduce the known major approaches deployed for understanding and using the geometry for design.

## 4.1 Differential Geometric Analysis

The most fundamental result starts with a differential geometric approach, i.e., understanding the local behavior of the surface. Huffman [27] describes the local behavior of a crease by introducing spherical trigonometry on the Gauss sphere, and this publication still remains a main reference. Resch investigated curved creases and stipulated that every space curve can be used to construct three distinct curved creases. He demonstrated such designs by early computer graphics rendering [16, 28]. Fuchs and Tabachnikov [29] further Huffman's work and contributed significantly to the understanding of developable surfaces and curved creases.. They assess that it is possible to fold an arbitrary curve drawn on paper into a 3D crease with higher curvature. If the curve is strictly convex and closes onto itself (e.g., circle) then the folded 3D crease is not in a plane. They also elaborate on the behavior of rulings along folded creases. On the other hand, Demaine et. al. [30] describe how paper behaves between creases and mathematically answer why only curved creases can produce interesting curved surface. In other words, the surface surrounded by straight creases cannot bend and must stay polyhedral. Even though these local analyses form the base of other geometric design approaches, these general results themselves stop at the first crease, while multiple creases are applied in practical designs.

## 4.2 Constructive Geometric Approach

One of the simplest design methods of curved folding with more than one crease is to use reflection. We start from a single developable surface and cut and reflect it by planes. The reflection is a special case of curved folding where the crease lies on a single osculating plane as described in [26]. This is a simple yet effective method and has been deployed by many artists. For example, Huffman's cone model is created using a single cone and its mirror reflections (see the reconstruction process in [31]).

## 4.3 Inverse Calculation of a Crease

More advanced methods solve an inverse problem to connect known elements such as cones and cylinders with a curved crease. Geretschläger [32] sets out to understanding curved creases by predefining the geometry of a piece of paper in a curved state. He then assumes the path of a crease and calculates the position of the part of the paper on the other side of the fold.

This type of inverse approach is useful to construct reusable modules for constructing symmetrically aligned or tessellated figures. Mosely analyzes curved creases of her own "cube shape", a volumetric model she invented that uses several tiles of a simple crease pattern made of four semicircles [33]. Also, in her tessellation works, a curve is numerically calculated so that cones and cylinders symmetrically tessellate a plane [34].

## 4.4 Discrete Geometric Approach

In order to deal with fully generalized curved folding without predetermined assumptions on the form of surfaces, we need to globally solve geometric problems by discretizing and globally solving the geometric problems.

Lalvani uses genetic algorithms that select mutated straight polyhedra. His "Morphological Genome Project" is based on defining parameter sets or genes, which are then used to modify a polyhedral shape. The final selected results by the genetic algorithm are developable [26]. The practical application of this work lead to metal column covers for interiors as described in Section 3.3.

Kergosien et al. [36] take an engineering approach to investigate early simulations of paper. Starting from a generic curve they are able to fit a developable surface. If the boundary curve creates crossing rule lines their algorithm finds a curved crease within the boundary as seen in.

Kilian et al. [35] model curved folding using planar quadrangle meshes (PQ-meshes) and deploy an optimization based method. A case study they investigated is a car design by Gregory Epps that is made of a single piece of paper. The physical model is 3D scanned and an elaborate process of analysis, rule line searching, plane fitting and edge optimization follows that results in a description of the piecewise developable surface. The work can post-rationalize a scanned paper model, which is useful for fabrication for instance, but does not describe the folding process or generate novel forms, which still presents the main challenge today.

## 5. Conclusion

We have reviewed a small portion of previous works of curved folding in art and design, examples of industrial applications of curved folding, and mathematical and computational considerations reflected in some of the designs. We hope that this study helps the development of novel curved folding design in a structural context.

## References


[1] MIURA K., “Proposition of pseudo-cylindrical concave polyhedral shells”, *Proc. IASS Symposium on Folded Plates and Prismatic Structures*, 1970.

[2] RESCH R. D. and CHRISTIANSEN H., “The design and analysis of kinematic folded plate systems”, *Proc. IASS Symposium on Folded Plates and Prismatic Structures*, 1970.

[3] TACHI T. and EPPS G., “Designing One-DOF Mechanisms for Architecture by Rationalizing Curved Folding”, in Proceedings of ALGODE 2011, 2011.

[4] SALLAS J., *Napkin folding, Vienna Imperial Furniture Collection and the Silver Collection*, (self published) 2010.

[5] JACOBSEN L. J., *Fra fladt papir til foldet lampeskærm*, Kunstuff, Danish Craft and Design, 2008.

[6] KLINT P. V. J., *P. V. Jensen Klint / Le Klint lighting*, 1943, `http://www.ylighting.com/leklint.html`

[7] ADLER E. D., *"A New Unity!" The Art and Pedagogy of Josef Albers*, University of Maryland, 2004.

[8] WINGLER H., *Bauhaus: Weimar, Dessau, Berlin, Chicago*, The MIT Press, 1978.

[9] LONDENBERG, K., *Papier und Form*, *Design in der Papierverarbeitung* , Krefeld: Scherpe, 1963. pp. 51-55.

[10] MCPHARLIN P., *Paper sculpture: its construction & uses for display & decoration* Marquardt & Company, incorporated, 1944.

[11] YENN T., “The story behind the Big Bang,” `http://erikdemaine.org/thok/parabel.html`.

[12] KASAHARA K., *Extreme Origami*, 1st ed., New York: Sterling, 2003, pp. 9-15.

[13] DEMAINE E. D. and DEMAINE. M. L., “Mathematics Is Art,” in *Proceedings of 12th Annual Conference of BRIDGES: Mathematics*, *Music, Art, Architecture, Culture*, Banff, Alberta, Canada, 2009, pp. 1–10.

[14] KOSCHITZ D., DEMAINE E. D., and DEMAINE M. L., “Curved Crease Origami,” in *proceedings of Advances in Architectural Geometry 2008*, Vienna, Austria, 2008, pp. 29–32.

[15] OGAWA H., *Forms of Paper*, Van Nostrand Reinhold (Trade), 1972.

[16] RESCH, R. BARNHILL R. E., and RIESENFELD R. F., “The Space Curve as a Folded Edge,” in *Computer-Aided Geometric Design*, Academic Press, Inc., 1974, pp. 255-258.

[17] WERTHEIM, M., “Cones, Curves, Shells, Towers: He Made Paper Jump to Life”, *The New York Times*, June 22, 2004.

[18] NEWMAN T. R., NEWMAN J., and NEWMAN L., *Paper As Art and Craft: The Complete Book of the History and Processes of the Paper Arts*, New York: Crown Publishers, 1973, p. 43.

[19] SCHMIDT P. and STATTMANN N., *Unfolded: Paper in Design, Art, Architecture and Industry*, 1st ed., Birkhäuser Architecture, 2009, p. 156, 241.

[20] SMITH, R., *Paper: Tear, Fold, Rip, Crease, Cut*, Black Dog Publishing, 2009, pp. 6-22.

[21] THOMAS J. and JACKSON P., *On Paper: New Paper Art,* illustrated edition, Merrell Holberton, 2001, p. 62.

[22] STERNBERG S., “Curves and Flats,” in *Origami$^4$*, A K Peters Ltd., 2009, pp. 9-20.

[23] IWAKI T. R., *The Mask Unfolds, Cavex Round Folding*, Artisans Gallery, 2010, p. 1.

[24] Jean-Charles Trebbi, L'art du pli - Design et décoration (Editions Alternatives, 2008): p. 43, 51, 95.

[25] SCHEIN M., “Applied Generative Procedures in Furniture Design,” in *Proceedings of the 5th International Conference GA 2002*, 2002.

[26] LALVANI H., “Bend the Rules of Structure” *Metropolis Magazine*, June 2003, http://www.metropolismag.com/html/content_0603/mgo/.

[27] HUFFMAN D. A.. “Curvature and creases: A Primer on Paper”. *IEEE Transactions on Computers*, Vol. C-25, No. 10, pp. 1010–1019, 1976.

[28] RESCH R. D., “Portfolio of Shaded Computer Images”, Proc. IEEE, Vol. 62, No. 4, 1974, pp.496-502.

[29] FUCHS D. and TABACHNIKOV S., “More on Paperfolding”. *The American Mathematical Monthly*, Vol. 106, No. 1, pp. 27–35, 1999.

[30] DEMAINE E. D., DEMAINE M. L., HART V., PRICE G. N., and TACHI T., (Non)existence of Pleated Folds: How Paper Folds Between Creases”, *Graphs and Combinatorics*, to appear.

[31] DEMAINE E. D., DEMAINE M. L. and KOSCHITZ D., “Reconstructing David Huffman's Legacy in Curved-Crease Folding”, *Origami$^5$: Proc. OSME 2010*, Singapore, July 13–17, 2010, to appear, A K Peters.

[32] GERETSCHLÄGER R., “Folding Curves,” in *Origami$^4$*, A K Peters Ltd, 2009: 151-164.

[33] MOSELY J. “The validity of the Orb, an Origami Model,” in Third International Meeting of Origami Science, Mathematics, and Education , AK Peters, Ltd., 2002, 75-82.

[34] MOSELY J. “Curved Origami,” in ACM SIGGRAPH 2008 Art & Design Galleries Catalog, 2008, 60-61.

[35] KILIAN M., FLÖRY S., MITRA N. J., and POTTMANN H., “Curved folding”, *ACM Transactions on Graphics*, Vol. 27, No. 3, 2008, pp. 1-9.

[36] KERGOSIEN Y., GOTODA H., and KUNII T., “Bending and Creasing Virtual Paper”, *IEEE Computer Graphics and Applications*, Vol. 14, No. 1, pp. 40–48, 1994.