MAS 964: The Nature of Mathematical Modeling - HW1

Sam Wolk

February 2023

1 Damped, Driven Harmonic Oscillator

In the following section, we will be studying a damped, (sometimes) driven harmonic oscillator, which is found by satisfying the following linear ordinary differential equation (unforced and forced):

$$m\ddot{x} + \gamma\dot{x} + kx = 0 \tag{1}$$

$$m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega t} \tag{2}$$

For simplicity, we will assume, $m > 0$ as it generally represents mass. We will also assume $k > 0$. These are reasonable assumptions because if $k < 0$ or $m < 0$, the same behavior can be found (including runaway amplification and over damping) subject to keeping track of sign flips. Rearranging the homogeneous equation (1) to eliminate the leading coefficient will also be useful:

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = 0 \tag{3}$$

Letting $\gamma_0 = \frac{\gamma}{m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$ in order to lighten the symbol pushing load, eq. (3) becomes the following:

$$\ddot{x} + \gamma_0\dot{x} + \omega_0^2 x = 0 \tag{4}$$

1.1 Simple Undamped Harmonic Motion

For an arbtrirary potential function $U(x)$, the force F exerted on a particle at location x is given by $F(x) = -\dot{U}(x)$. Since $F(x) = m\ddot{x}$ by Newton's second law, this means that $m\ddot{x} = -\dot{U}(x)$.

For undamped harmonic motion, the position $x(t)$ must satisfy $m\ddot{x} = -kx$, and as such, $\dot{U}(x) = kx$ must be true. Integrating both sides, we get $U(x) = \frac{1}{2}kx^2 + c$ - in other words, the potential energy of the system must increase quadratically around the potential minimum c.

We could also look at the Taylor expansion of $\dot{U}(x)$ about $x = 0$:

$$\dot{U}(x) = \dot{U}(0) + \ddot{U}(0)x + \dddot{U}(0)\frac{x^2}{2} + ... = kx \tag{5}$$

Since the Taylor expansion must simplify to kx, we see that $\dot{U}(0) = 0$, and so the force exerted on the particle vanishes at $x = 0$. Furthermore, the second derivative of the potential energy function at $x = 0$ is equal to the equivalent of the spring constant, i.e. $\ddot{U}(0) = k$. Additionally, we see that even if the third derivatives and beyond do not vanish at zero, there will be some $-\epsilon < x < \epsilon$ for which the system effectively behaves as a harmonic oscillator. We can also perform the Taylor expansion on $U(x)$ instead of $\dot{U}(x)$ to reach the same conclusions:

$$U(x) = U(0) + \dot{U}(0)x + \ddot{U}(0)\frac{x^2}{2} + ... = \frac{1}{2}kx^2 + c \tag{6}$$

1.2 Solving the Homogeneous Equation

Taking an *ansatz* of $x_p = Ae^{rt}$, by Eqs. 4.5 and 4.6 in NMM, the solution to the homogeneous linear ODE is determined by solving the characteristic equation:

$$r^2 + \gamma_0 r + \omega_0^2 = 0 \tag{7}$$

As eq. (7) is a simple quadratic equation, r can be found by the quadratic formula:

$$r = \frac{-\gamma_0 \pm \sqrt{\gamma_0^2 - 4\omega_0^2}}{2} \tag{8}$$

As such, we get an equation for the general solution:

$$\begin{aligned} x_g &= \sum_{j=1}^{2} C_j A e^{r_j t} \\ r_j &= \frac{-\gamma_0 + (-1)^j \sqrt{\gamma_0^2 - 4\omega_0^2}}{2} \end{aligned} \tag{9}$$

Plugging r back into the *ansatz* x_p allows us to better understand the relationship between the parameters and motion.

$$\begin{aligned} x_p &= Ae^{\frac{-\gamma_0}{2}t \pm \frac{\sqrt{\gamma_0^2-4\omega_0^2}}{2}t} \\ &= Ae^{\frac{-\gamma_0}{2}t} e^{\frac{\pm\sqrt{\gamma_0^2-4\omega_0^2}}{2}t} \end{aligned} \tag{10}$$

1.2.1 Coditions for Undamped Oscillation

We can see from the paticular solution in (10) then that the equation is the product of two exponentials, and so for undamped harmonic motion to occur, the first condition is that $\gamma_0 = 0 \implies \gamma = 0$, i.e. the velocity does not affect the kinematics (no friction). Substituting $\gamma_0 = 0$ into x_p gives the following equation which illustrates the undamped harmonic oscillation:

$$x_p = Ae^{\pm i\omega_0 t} = A\cos(\omega_0 t) \pm iA\sin(\omega_0 t) \tag{11}$$

As such, the frequency of oscillation will be $\frac{\omega_0}{2\pi}$ rads/unit time, which is equal to $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$, and the amplitude A is independent of the system parameters m, γ, and k, depending solely on the initial condition $x(0) = Ae^0 = A$.

1.2.2 Conditions for Damped Oscillation

In order for damped oscillation to occur (so $\gamma_0 \neq 0 \implies \gamma \neq 0$), r must still have an imaginary component, and so $\gamma_0^2 < 4\omega_0^2$. Simplifying gives conditions for γ, k, m:

$$\begin{aligned} \gamma_0^2 &< 4\omega_0^2 \\ \frac{\gamma^2}{m^2} &< 4\frac{k}{m} \\ \gamma^2 &< 4km \end{aligned} \tag{12}$$

Runaway amplification will occur if $\gamma < 0$, since this implies that $\gamma_0 < 0$ and so $-\frac{\gamma_0}{2} > 0$, meaning that as $t \to \infty$, so too does $e^{-\frac{\gamma_0}{2}t} \to \infty$ according to (10). As such, we will focus on $\gamma > 0$. When $\gamma > 0$, and condition (12) holds, then the behavior is damped oscillation, i.e. equation (11) multiplied by an exponential decay envelope.

1.2.3 Conditions for Over-damping

If the damping condition (12) does not hold, then, no oscillation will occur, because each r_j has no imaginary component.

For instance, when $\gamma^2 = 4km$ (i.e. $\gamma_0 = 2\omega_0$), then $x_p = Ae^{-\frac{\gamma_0}{2}t} = e^{-\omega_0 t}$, showing a direct relationship between the decay constant ω_0 and the the natural resonant frequency of the oscillator with the given m and k values.

Alternatively, when $\gamma^2 > 4km$ (i.e. $\gamma_0^2 > 4\omega_0^2$, the position will simply be an exponential decay since both r_1 and r_2 will be negative as $\sqrt{\gamma_0^2 - 4\omega_0^2}$ is necessarily less than γ_0 assuming given that $k \neq 0$.

1.3 Driven Oscillator Frequency Response

For a harmonic oscillator driven by a sinusoidal forcing function $e^{i\omega t}$, intuition suggests that the system will behave as a filter of some sort, i.e. the output will be a scaled copy of the input sinuosoid, with the scaling factor depending on the input signal's frequency, and also subject to a possible phase shift. As such, we will take an *ansatz* of $x(t) = Ae^{i\omega t}$, and attempt to show that this satisfies the inhomegeneous equation (2) and then find the relationship between A and ω. First we find the derivatives of $x(t)$:

$$\dot{x} = Ai\omega e^{i\omega t} \qquad \qquad \ddot{x} = -A\omega^2 e^{i\omega t}$$

Next, we populate equation (2) and simplify:

$$\begin{aligned} e^{i\omega t} &= -mA\omega^2 e^{i\omega t} + \gamma Ai\omega e^{i\omega t} + kAe^{i\omega t} \\ 1/A &= -m\omega^2 + \gamma i\omega + k \\ 0 &= -m\omega^2 + \gamma i\omega + k - 1/A \end{aligned} \tag{13}$$

As A is allowed to be complex, this equation will always have a solution. Re-arranging equation (13) gives A in terms of ω:

$$\begin{aligned} A(\omega) &= \frac{1}{-m\omega^2 + \gamma i\omega + k} \\ &= \frac{1}{k - m\omega^2 + \gamma i\omega} \\ &= \frac{k - m\omega^2 - \gamma i\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2} \end{aligned} \tag{14}$$

$$Re[A] = \frac{k - m\omega^2}{(k - m\omega^2)^2 + \gamma^2\omega^2} \qquad \qquad Im[A] = -\frac{\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2} \tag{15}$$

1.3.1 Undamped Driven Harmonic Oscillator

When $\gamma = 0$ (or is otherwise small relative to ω), we see that the imaginary part disappears, leaving us with $A_{\gamma=0} = Re[A] = \frac{1}{k - m\omega^2}$. This gives us the resonant frequency of the oscillator as before by finding the zeros of the denominator, i.e. $\omega = \sqrt{\frac{k}{m}}$. If there is no damping, then hitting precisely this frequency blows up A to infinity - the proverbial glass shatters!

1.3.2 Damped Driven Harmonic Oscillator

When γ is not small, then $A = Re[A] + Im[A]i = a + bi$, and we see that the output of the system is in fact the sum of two sinusoids which are out of phase but with the same frequency:

$$
\begin{aligned}
x(t) &= Ae^{i\omega t} \\
&= ae^{i\omega t} + bie^{i\omega t} \\
&= a\sin(\omega t) + ai\cos(\omega t) + bi\sin(\omega t) + bi^2\cos(\omega t) \\
&= a\sin(\omega t) + bi^2\cos(\omega t) + ai\cos(\omega t) + bi\sin(\omega t) && (16) \\
&= a\sin(\omega t) - b\cos(\omega t) + i(a\cos(\omega t) + b\sin(\omega t)) && (17) \\
&= a\sin(\omega t) - b\sin(\frac{\pi}{2} - \omega t) + i(a\cos(\omega t) + b\sin(\omega t)) && (18) \\
&= a\sin(\omega t) + b\sin(\omega t - \frac{\pi}{2}) + i(a\cos(\omega t) + b\sin(\omega t)) && (19)
\end{aligned}
$$

Intuitively then, we see that the real component of $x(t)$ is just a sinusoid at the same frequency as the original signal summed with a phase-shifted copy of itself, with each part of the sum subject to scaling. As such, the relative balance of the real and imaginary parts of A (a and b respectively) determine the amount of phase cancellation resulting from adding the original signal to a delayed copy of itself, i.e. the balance between a and b determine the gain of the harmonic oscillator's response to the input signal. Similarly, by looking at the imaginary component of x(t), we see that the phase of the output in relation to the input will change depending on the balance between a and b.

Alternatively, we can rewrite equation (13) in polar form by letting $\theta = \arctan(\frac{\gamma\omega}{k-m\omega^2})$ and $d = \sqrt{(k-m\omega^2)^2 + \gamma^2\omega^2}$:

$$
\begin{aligned}
1/A &= d\cos(\theta) + id\sin(\theta) \\
&= de^{i\theta} \\
A &= \frac{1}{d}e^{-i\theta} && (20)
\end{aligned}
$$

Finally, we can write $x(t)$ purely in terms of m, γ, k, and ω (subject to a few substitutions for brevity) by plugging equation (20) into our *ansatz*:

$$
\begin{aligned}
x(t) &= \frac{1}{d}e^{-i\theta}e^{i\omega t} \\
&= \frac{1}{d}e^{i(\omega t - \theta)} && (21)
\end{aligned}
$$

At long last, we have an explicit description of the system response: a sinusoid at the same frequency as the input, phase-shifted by $\theta = \arctan(\frac{\gamma\omega}{k-m\omega^2})$ radians, with an amplitude of $\frac{1}{\sqrt{(k-m\omega^2)^2+\gamma^2\omega^2}}$. Whenever γ or ω are small, we recover the original phase, since $\theta = \arctan(0) = 0$.

To find the center frequency of the filtration action, we need to maximize the amplitude of the response, which will occur when the denominator of the amplitude expression $1/d$ is minimized. Assuming $\gamma \neq 0$ and γ is small, we see that this should be near $\omega_c = \sqrt{\frac{k}{m}}$, which is again the natural resonant frequency of the system found in previous steps.

An interactive plot can be found here, with ω and the γ as interactive sliders labeled w and a.

1.4 Q Factors

TODO

1.5 Laplace Transforms

Given that $x(0) = 0$ and $\dot{x}(0) = 0$, we can fairly easily solve the inhomogeneous differential equation using the Laplacian:

$$\begin{aligned}
\mathcal{L}\{m\ddot{x}\} &= m(s^2X(s) - sx(0) - \dot{x}(0)) = ms^2X(s) \\
\mathcal{L}\{\gamma\dot{x}\} &= \gamma(sX(s) - x(0)) = \gamma sX(s) \\
\mathcal{L}\{kx\} &= kX(s) \\
\mathcal{L}\{f(t)\} &= F(s)
\end{aligned}$$

Plugging these into equation 4.58 gives us the transformed differential equation:

$$\begin{aligned}
\mathcal{L}\{m\ddot{x}\} + \lambda\dot{x} + kx\} &= \mathcal{L}\{f(t)\} \\
ms^2X(s) + \gamma sX(s) + kX(s) &= F(s) \\
X(s) &= \frac{1}{(ms^2 + \gamma s + k)}F(s)
\end{aligned}$$

Letting $R(s) = \frac{1}{(ms^2+\gamma s+k)}$, we see that $X(s) = R(s)F(s)$, so $\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{R(s)F(s)\} = r(t) * f(t)$, in other words, an input signal convolved with an impulse response. If we let $f(s) = e^{i\omega t}$, we can find the frequency response of the system for a generic sinusoid. First we rewrite $R(S)$ using the roots of the quadratic equation of s in the denominator:

$$R(s) = \frac{1}{(s + \frac{\gamma - \sqrt{\gamma^2 - 4mk}}{2m})(s + \frac{\gamma + \sqrt{\gamma^2 - 4mk}}{2m})} \tag{22}$$

Next, we observe hat $\mathcal{L}\{e^{i\omega t}\} = \frac{1}{s - i\omega t}$, and so w get an equation for $X(s)$:

$$X(s) = \frac{1}{(s + \frac{\gamma - \sqrt{\gamma^2 - 4mk}}{2m})(s + \frac{\gamma + \sqrt{\gamma^2 - 4mk}}{2m})(s - i\omega t)} \tag{23}$$

Finally, we can invert back into the original space by using a Laplace-transform table, as this follows the pattern $x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{\frac{1}{(s-a)(s-b)(s+c)}\} = \frac{ae^{-at}}{(a-b)(a-c)} + \frac{be^{-bt}}{(b-a)(b-c)} + \frac{ce^{-ct}}{(c-a)(c-b)}$. Using $a, b = \frac{\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$ and $c = i\omega t$, we now have an explicit expression for $x(t)$, albeit one that is devilish to simplify... it will be left as an exercise for the reader :)

1.6 Non-linear Corrections

TODO

2 Coupled Harmonic Oscillators

Assume x_1 and x_2 are both measured separately where 0 corresponds to the respective potential minimum. If mass 2 is held fixed, and then mass 1 is moved to from its position of 0 to x_1, then the force acting on mass 1 will be $2kx_1$ - half contributed by the spring on the left pulling it back, and half contributed by the spring on the right pushing it back to the minimum. If mass 2 on the other hand moves to x_2 and is fixed there, it will result in a force of $-kx_2$ acting on mass 1.

We can then write the differential equation for each mass:

$$\begin{aligned}
\ddot{x}_1 - 2kx_1 + kx_2 = 0 \\
\ddot{x}_1 - 2kx_2 + kx_1 = 0
\end{aligned} \tag{24}$$

The coupling matrix $\mathbf{A}$ is then trivial to write:

$$\mathbf{A} = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}$$

Stack the positions and forces into column vector:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \ddot{\mathbf{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}$$

We can now rewrite equation (24):

$$\ddot{\mathbf{x}} + \mathbf{A} \cdot \mathbf{x} = \mathbf{0} \tag{25}$$

Now we find the eigenvectors in order to describe the change of variables for representing the normal modes. First, we find the characteristic polynomial of $\mathbf{A}$:

$$det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -2k-\lambda & k \\ k & -2k-\lambda \end{vmatrix} = 4k^2 + 4k\lambda + \lambda^2 - k^2 = 3k^2 + 4k\lambda + \lambda^2 = (3k+\lambda)(k+\lambda) \tag{26}$$

Setting the characteristic polynomial to 0 let's us find the eigenvalues, $\lambda = -3k, -k$. Plugging these back in to the eigenvector/value identity lets us solve for the eigenvectors:

$$(\mathbf{A} + k\mathbf{I})\mathbf{v} = \mathbf{0}$$
$$\begin{bmatrix} -2k+k & k \\ k & -2k+k \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigenvector $\mathbf{v}$ must satisfy the equations below.

$$-kv_1 + kv_2 = 0$$
$$kv_1 - kv_2 = 0$$

Let $v_1 = 1$, so $v_2 = 1$. One eigenvector is then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$(\mathbf{A} + 3k\mathbf{I})\mathbf{v} = \mathbf{0}$$
$$\begin{bmatrix} -2k+3k & k \\ k & -2k+3k \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigenvector $\mathbf{v}$ must satisfy the equations below.

$$kv_1 + kv_2 = 0$$
$$kv_1 + kv_2 = 0$$

Let $v_1 = 1$, so $v_2 = -1$. The other eigenvector is then $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The matrix of eigenvectors can now be written:

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{27}$$

Now, we can perform our change of variables. To do so, we first invert $\mathbf{M}$:

$$\begin{aligned}
\mathbf{M}^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\
&= -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

This finally yields our system subject to the change of variables:

$$\begin{aligned}
\mathbf{z} = \mathbf{M}^{-1}\mathbf{x} &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x_1+x_2) \\ \frac{1}{2}(x_1-x_2) \end{bmatrix} \\
\mathbf{D} = \mathbf{M}^{-1}\cdot\mathbf{AM} &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -k & -3k \\ -k & 3k \end{bmatrix} \\
&= \begin{bmatrix} -k & 0 \\ 0 & -3k \end{bmatrix}
\end{aligned}$$

As expected, the new coupling matrix has the eigenvalues on the diagonal.

We can now interpret the system's behavior for the normal modes intuitively. First, we observe that the two eigenvectors (concatenated into $\mathbf{M}$) essentially correspond to moving the original masses by the same amount in either the same direction or in opposite directions. Next, looking at $\mathbf{z}$ we can confirm that in the change of variables, when the motions described above occur, either $z_1 = 1, z_2 = 0$ or $z_1 = 0, z_2 = 1$ depending on whether or not the initial displacement was in the same direction or opposite directions, and as such, multiplying by the new coupling matrix will pick out either the first or second eigenvalue accordingly, resulting in a single undamped harmonic oscillator differential equation of either z_1 or z_2 (but not both). At the same time, when these conditions, it's easy to confirm by inspection that the original masses will also be exhibiting undamped simple harmonic motion (either in the same direction or opposite directions but with the same frequency). $\mathbf{z}$ represents two separate oscillatory behaviors of the system *as a whole.* which are always occurring independently and super imposed upon one another - just the sum of the two modes, and so when the amplitude of one of the modes vanishes, the other emerges plainly. As the amplitude of one of the modes increases the masses begin to demonstrate beating phenomenon - very cool!

2.1 A note on Coupled Oscillators in Audio Synthesis

there is an audio synthesis technique from the world of computer music/modern modular synthesizers called "Scanned Synthesis" developed by Max Matthews et al in the late 90s. In this system (usually programmed digitally, but I suppose you could do it with analog electronics theoretically), you setup a series of masses in a line, each connected by a spring to the adjacent masses. You then excite the system by setting starting positions of each mass. The masses/spring constants are tuned so that the normal modes of the system are at low frequencies (e.g. 0.1 Hz to 20 Hz). If the normal modes were higher (i.e. in the audible range) you could of course auralize the position of each mass directly, but this would be sightly boring - it would just be a single, likely dissonant chord or tone, but possibly exhibiting some interesting beating phenomenon. Instead, with the normal modes in the low frequency range, you treat x as a wavetable - in other words, you write the current positions of each mass into a buffer sequentially, and then use that buffer as the waveform for a new, separate audio rate oscillator which you run at any frequency (i.e. pitch) that you like (typically interpolating samples between the mass positions, i.e. if you have 8 masses in the system, you might smooth their positions out into a 50 sample buffer with some curvature instead of sharp angles). The changing positions of the masses over time corresponds to the audio-rate oscillator's waveform continuously but smoothly changing over time, resulting in a very rich, spectrally complex timbre which is constantly evolving. In the example from the homework, with two masses, you are effectively limited to creating triangle wave-like shapes (subject to some smoothing if desired). Interestingly, when you use the first eigenvector (with the masses displaced in the same direction) as the starting positions resulting in the symmetric mode, this would correspond to no audio output, effectively DC, since the positions are always identical and the normal mode is in the sub-audio range by design, but you could use the slow oscillation output as an LFO control source for another parameter (with frequency determined by the eigenvalue). When the other eigenvector is used for the initial displacement, the positions of the masses are always equal but opposite, so your audio-rate oscillator's waveform would be a perfect triangle wave (or sine wave if lightly interpolated) centered on 0 which changes in amplitude with frequency according to the eigenvalue- effectively a tremolo affect applied to the oscillator! So no timbral modulation, purely amplitude. To make things more interesting, scanned synthesis implementations will usually include stiffness, mass, and damping as modulatable parameters of the system, allowing you to alter the eigenvectors/values and normal modes in real time. Additionally, they may allow you to trigger an impulse to the system (with controllable shape) on demand, allowing you to create percussive effects when damping is high or temporary deviations from the timbral modulation pattern when damping is low on demand. Some even let you send an audio-rate signal through the system as a forcing signal! My favorite implementation is the "Scanned" module made by Qu-Bit Electronix in nyc.

3 Digital Filters

Intuitively, the digital filter below is a lagging integrator which will smooth out high-frequency components, as it is a weighted average which crossfades between the incoming sample from the forcing signal and the previous output signal as α ranges over $[0, 1]$.

$$y(k) = \alpha y(k-1) + (1-\alpha)x(k) \tag{28}$$

To find the frequency response of the filter, we will use $\mathcal{Z}$-transforms, taking $y(k < 0) = 0$:

$$\begin{aligned}
\mathcal{Z}\{y(k)\} &= Y(z) \\
\mathcal{Z}\{\alpha y(k-1)\} &= \alpha(z^{-1}Y(z) - y(-1)) = \alpha z^{-1}Y(z) \\
\mathcal{Z}\{(1-\alpha)x(k)\} &= (1-\alpha)X(z) \\
Y(z) &= \alpha z^{-1}Y(z) + (1-\alpha)X(z) \\
Y(z) - \alpha z^{-1}Y(z) &= (1-\alpha)X(z) \\
Y(z)(1-\alpha z^{-1}) &= (1-\alpha)X(z) \\
H(z) &= \frac{Y(z)}{X(z)} = \frac{1-\alpha}{1-\alpha z^{-1}} = \frac{z - z\alpha}{z-\alpha} = \frac{z}{z-\alpha} - \alpha\frac{z}{z-\alpha} \\
h(k) &= \alpha^k(1-\alpha) = \alpha^k - \alpha^{k+1}
\end{aligned}$$

Convolving $h(x)$ with an input signal gives the output of the system - intuitively, this is the saying that for each sample, at timestep k steps after the sample first appears, the current sample will still appear in the output but scaled by an exponential decrease of α^k - which corresponds with the naive interpretation of the difference equation. Plugging $e^{j\omega}$ into $H(z)$ gives the transfer function of a generic sinusoidal input signal:

$$H(e^{i\omega}) = \frac{e^{i\omega}}{e^{i\omega} - \alpha} - \alpha\frac{e^{i\omega}}{e^{i\omega} - \alpha}$$

$$
\begin{aligned}
A &= \frac{e^{i\omega}}{e^{i\omega} - \alpha} \\
&= \frac{e^{i\omega}}{e^{i\omega} - \alpha} \frac{e^{-i\omega} - \alpha}{e^{-i\omega} - \alpha} \\
&= \frac{1 - \alpha e^{i\omega}}{1 + \alpha^2 - \alpha e^{i\omega} - \alpha e^{-i\omega}} \\
&= \frac{1 - \alpha e^{i\omega}}{1 + \alpha^2 - \alpha i \sin(\omega) - \alpha \cos(\omega) + \alpha i \sin(\omega) - \alpha \cos(\omega)} \\
&= \frac{1 - \alpha e^{i\omega}}{1 + \alpha^2 - 2\alpha \cos(\omega)} \\
&= \frac{1 - \alpha e^{i\omega}}{\alpha^2 - 2\alpha \cos(\omega) + \cos^2(\omega) + 1 - cos^2(\omega)} \\
&= \frac{1 - \alpha e^{i\omega}}{\alpha^2 - 2\alpha \cos(\omega) + \cos^2(\omega) + sin^2(\omega)} \\
&= \frac{1 - \alpha e^{i\omega}}{(\alpha + \cos(\omega))^2 + sin^2(\omega)} \\
&= \frac{1 - \alpha \cos(\omega) - \alpha i \sin(\omega)}{(\alpha + \cos(\omega))^2 + sin^2(\omega)}
\end{aligned}
$$

$$
\begin{aligned}
Re[A] &= \frac{1 - \alpha \cos(\omega)}{(\alpha + \cos(\omega))^2 + sin^2(\omega)} \\
Im[A] &= \frac{-\alpha \sin(\omega)}{(\alpha + \cos(\omega))^2 + sin^2(\omega)} \\
H(e^{i\omega}) &= (1 - \alpha)A
\end{aligned}
$$

Finding the amplitude of the frequency response can now be completed by taking the magnitude of $H(e^{i\omega})$, as the transfer function is fully defined in terms of ω and α.